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PAPER III

REAL ANALYSIS

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PAPER III : REAL ANALYSIS

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Chapter 1

THE REAL AND COMPLEX NUMBER SYSTEMS

We know that $\square$, the set of all integers, is integral domain and $\square$, the set of rational numbers, is the smallest field containing $\square$ (Ref. pp38: Chapter 8 SDE Study Material *Linear Algebra*). But the rational number system is inadequate for many purposes, both as a field and as an ordered set. For instance, next theorem says that, there is no rational p such that $p^2 = 2$. This leads to the introduction of so-called "irrational numbers" which are often written as infinite decimal expressions and are considered to be "approximated" by the corresponding finite decimals. Thus the sequence

$$1, 1.4, 1.414, 1.4142, \ldots$$

"tends to $\sqrt{2}$." But unless the irrational number $\sqrt{2}$ has been clearly defined, the question must arise: Just what is it that this sequence "tends to"?

This sort of question can be answered as soon as the so-called "real number system" is constructed.

In the coming two theorems we show that the rational number system has certain gaps, in spite of the fact that between any two rational there is another: If $r < s$, then $r < \frac{r+s}{2} < s.$ The real number system fills these gaps. This is the principal reason for the fundamental role which it plays in analysis.

(1.1) Theorem The equation $p^2 = 2$ has no solutions in rational numbers. Thus $\sqrt{2}$ is not a rational number.

Proof

Suppose that $p = m/n$ for $m, n \in \mathbf{Z}$ is a rational number such that $p^2 = 2$. We assume that we have cancelled any factors common to m and n, so that the integers m and n are are not both even. Then $p^2 = 2$ implies

$$m^2 = 2n^2, \qquad \ldots (1)$$

and shows that that m^2 is an even integer, which implies that m is also even (For, otherwise m is odd implies $m = 2k+1$ for some integer k implies $m^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1,$ which is also odd, contradicting the fact that m^2

is an even integer). Hence m^2 is divisible by 4. It follows from (1) that $2n^2$ is also divisible by 4 or n^2 is divisible by 2 or n^2 is even. Hence by the argument above, n is even. Hence we arrives at the conclusion that both m and n are even, contrary to our assumption that m and n are not both even. Hence $p^2 = 2$ is not satisfied by a rational number p.

(1.2) Theorem

(i) $A = \{p \in \square : p > 0, p^2 < 2\}$ has no largest number in $\square$.

(ii) $B = \{p \in \square : p > 0, p^2 > 2\}$ has no smallest number in $\square$.

Proof

To prove the above, we associate with each rational $p > 0$ the number

$$q = p - \frac{p^2 - 1}{p + 2} = \frac{2p + 2}{p + 2}. \qquad \ldots (2)$$

Then

$$q^2 - 2 = \frac{2\left(p^2 - 2\right)}{\left(p + 2\right)^2}. \qquad \ldots (3)$$

If $p \in A,$ then $p^2 - 2 < 0,$ hence first equality in (2) shows that $q > p$, and (3) shows that $q^2 < 2.$ Thus $q \in A.$

If $p \in B,$ then $p^2 - 2 > 0,$ hence first and second equalities in (2) shows that $0 < q < p$, and (3) shows that $q^2 > 2.$ Thus $q \in B.$

2 Ordered Sets

(2.1) Definition Let S be a set. An **order** on S is a relation, denoted by <, with the following two properties:

(i) If $x \in S$ and $y \in S$ the one and only one of the statements

$$x < y, \quad x = y, \quad y < x$$

is true.

(ii) If $x, y, z \in S$, if $x < y,\ y < z$, then $x < z.$

(2.2) Definition An **ordered set** is a set S in which an order is defined.

Example: Show that $\mathbb{Q}$ is an ordered set if $r < s$ is defined to mean that $s - r$ is a positive rational number.

Solution

(i) If $r \in \mathbb{Q}$ and $s \in \mathbb{Q}$ then either $s - r$ is a positive rational number or $s - r = 0$ or $r - s$ is positive number. i.e., one and only one of the statements $x < y,\ x = y,\ y < x$ hold.

(ii) If $r, s, t \in \mathbb{Q}$, and if $r < s,\ s < t$, then $s - r$ is a positive rational number and $r - t$ is a positive rational number *implies* $s - t = (s - r) + (r - s)$ is a positive rational number *implies* $r < t$.

(2.3) Definition Suppose S is an ordered set, and $E \subset S$. If there exists a $\beta \in S$ such that $x \leq \beta,\ \forall x \in E,$ we say that E is **bounded above,** and call β an **upper bound** of E.

(2.4) Definition Suppose S is an ordered set, $E \subset S,$ and E is bounded above. Suppose there exists an $\alpha \in S$ with the following properties:

(i) α is an upper bound of E.

(ii) If $\gamma < \alpha$ then γ is not an upper bound of E.

Then α is called the **least upper bound** (or **supremum**) of E, and we write

$$\alpha = \sup E.$$

(2.5) Definition Suppose S is an ordered set, and $E \subset S$. If there exists a $\beta \in S$ such that $x \leq \beta,\ \forall x \in E,$ we say that E is **bounded below,** and call β a **lower bound** of E.

(2.6) Definition Suppose S is an ordered set, $E \subset S,$ and E is bounded below. Suppose there exists an $\alpha \in S$ with the following properties:

(iii) α is a lower bound of E.

(iv) If $\gamma > \alpha$ then γ is not a lower bound of E.

Then α is called the **greatest lower bound** (or **infimum**) of E, and we write

$$\alpha = \inf E.$$

(2.7) Examples

***Example* 1** Consider the sets A and B [in Theorem (1.2)] as subsets of the ordered set $\mathbb{Q}$. The set A is bounded above. In fact, the upper bounds of A are exactly the members of B. Since B contains no smallest element, A has no least upper bound in $\mathbb{Q}$.

Similarly, B is bounded below: The set of all lower bounds of B consists of A and of all $r \in \square$ with $r \leq 0$. Since A has no largest element, B has no greatest lower bound in $\square$.

***Example* 2** If α = sup E exists, then α may or may not be a member of E. For example, let $E_1 = \{r \in \square : r < 0\}$ and $E_2 = \{r \in \square : r \leq 0\}$. Then

$$\sup E_1 = \sup E_2 = 0,$$

and $0 \notin E_1, 0 \in E_2$.

***Example* 3** Let $E = \left\{\frac{1}{n} : n \in \square\right\}$. Then $\sup E = 1 \in E$ and $\inf E = 0 \notin E$.

(2.8) Definition An ordered set S is said to have the **least upper bound property** if the following is true:

If $E \subset S$, $E \neq \Phi$ and E is bounded above, then $\sup E$ exists in S.

(2.9) Example Describe the least upper bound property of ordered sets. Verify whether the ordered field $\square$ of rationals have this property. (UQ 2008)

Solution

Consider the set $A = \{p \in \square : p > 0, p^2 < 2\}$ which is discussed in Theorem (1.2) and Example 1 in (2.7). The set A is bounded above. In particular, A is bounded above by each element of $B = \{p \in \square : p > 0, p^2 > 2\}$ discussed in Theorem (1.2). But there is no least upper bound for the set A as A has no largest element and B has no smallest number in $\square$, by Theorem (1.2), and $p \in \square$ with $p^2 = 2$ is not rational by Theorem (1.1). Hence:

$\square$ does not have the least upper bound property.

(2.10) Theorem Suppose S is an ordered set with the least-upper bound property, $B \subset S$, $B \neq \Phi$, and B is bounded below. Let L be the set of all lower bounds of B. Then

$$\alpha = \sup L$$

exists in S, and $\alpha = \inf B$.

In particular, $\inf B$ exists in S.

Proof Since B is bounded below, L is not empty. Since L consists of exactly those $y \in S$ which satisfy the inequality $y \leq x \ \forall x \in B$, we see that every $x \in B$ is an upper bound of L.

Thus L is bounded above. Since, by the hypothesis, S is an ordered set with least upper bound property, we obtain that L has supremum in S; call it α.

If $\gamma < \alpha$ then [by Definition (2.4)] γ is not an upper bound of L, hence $\gamma \notin B$. It follows that $\alpha \leq x \ \forall x \in B$. Thus $\alpha \in L$.

If $\alpha < \beta$ then $\beta \notin L$, since α is an upper bound of L.

We have shown that $\alpha \in L$ but $\beta \notin L$ if $\beta > \alpha$. In other words, α is a lower bound of B, but β is not if $\beta > \alpha$. This means that $\alpha = \inf B$.

3 FIELDS

(3.1) Definition A **field** is a set F with two operations, called *addition* and *multiplication*, which satisfy the following so-called "field axioms" (A), (M), and (D):

(A) Axioms for addition

(A1) If $x \in \mathsf{F}$ and $y \in \mathsf{F}$, then their sum $x + y \in \mathsf{F}$.

(A2) Addition is commutative: $x + y = y + x \ \forall \ x \in \mathsf{F}$ and $y \in \mathsf{F}$.

(A3) Addition is associative: $(x + y) + z = x + (y + z) \ \forall \quad x, y, z \in \mathsf{F}$.

(A4) F contains an element 0 such that $0 + x = x \ \forall x \in \mathsf{F}$.

(A5) To every $x \in \mathsf{F}$ corresponds an element $-x \in \mathsf{F}$ such that $x + (-x) = 0$.

(B) Axioms for multiplication

(A1) If $x \in \mathsf{F}$ and $y \in \mathsf{F}$, then their product $xy \in \mathsf{F}$.

(A2) Multiplication is commutative: $xy = yx \ \forall \ x \in \mathsf{F}$ and $y \in \mathsf{F}$.

(A3) Multiplication is associative: $(xy)z = x(yz) \ \forall \quad x, y, z \in \mathsf{F}$.

(A4) F contains an element $1 \neq 0$ such that $1x = x \ \forall x \in \mathsf{F}$.

(A5) If $x \in \mathsf{F}$ and $x \neq 0$ then there exists an element $\frac{1}{x} \in \mathsf{F}$ such that $x \cdot \frac{1}{x} = 1$.

(D) The distributive law

$$x(y + z) = xy + xz \ \forall \ x, y, z \in \mathsf{F}.$$

(3.2) Remarks

- One usually write (in any field)

$$x-y,\ \frac{x}{y},\ x+y+z,\ xyz,\ x^2,\ x^3,\ 2x,\ 3x,\ \ldots$$

in place of

$$x+(-y),\ x\cdot\left(\frac{1}{y}\right),\ (x+y)+z,\ (xy)z,\ xx,\ xxx,\ x+x,\ x+x+x,\ \ldots$$

(3.3) Examples

The field axioms clearly hold in if addition and multiplication have their customary meaning. Thus Q is a field. Similarly, R and C are fields. Z is not a field.

(3.4) Proposition The axioms for addition imply the following statements:

(a) (*Cancellation law*) If $x+y=x+z$ then $y=z$.

(b) (*Uniqueness of zero element*) If $x+y=x$ then $y=0$.

(c) (*Uniqueness of inverse element*) If $x+y=0$ then $y=-x$.

(d) $-(-x)=x$.

Proof

(*a*) Suppose $x+y=x+z$. Now

$$\begin{aligned} y &= 0+y, \text{ by (A4)} \\ &= (-x+x)+y, \text{ using (A2) and (A5)} \\ &= -x+(x+y), \text{ using (A3)} \\ &= -x+(x+z), \text{ using the assumption} \\ &= (-x+x)+z, \text{ using (A3)} \\ &= 0+z, \text{ using (A2) and (A5)} \\ &= z, \text{ using (A4)} \end{aligned}$$

(*b*) Take $z = 0$ in (*a*) above to obtain (*b*).

(*c*) Take $z = -x$ in (*a*) above to obtain (*c*).

(*d*) Since $-x+x=0$, by (*c*) above, taking $-x$ in place of x and x in place of y, we obtain $-(-x)=x$.

(3.5) Proposition The axioms for multiplication imply the following statements.

(a) (*Cancellation law*) If $x\neq 0$ and $xy=xz$ then $y=z$.

(b) (*Uniqueness of zero element*) If $x \neq 0$ and $xy = x$ then $y = 1$.

(c) (*Uniqueness of inverse element*) If $x \neq 0$ and $x + y = 1$ then $y = \frac{1}{x}$.

(d) If $x \neq 0$ then $\frac{1}{\frac{1}{x}} = x$.

Proof

(*a*) Suppose $x \neq 0$ and $xy = xz$. Now

$$\begin{aligned} y &= 1y, \text{ by (B4)} \\ &= \left(\frac{1}{x} \cdot x\right) y, \text{ using (B2) and (B5)} \\ &= \frac{1}{x}(xy), \text{ using (B3)} \\ &= \frac{1}{x}(xz), \text{ using the assumption} \\ &= \left(\frac{1}{x} \cdot x\right) z, \text{ using (B3)} \\ &= 1z, \text{ using (B2) and (B5)} \\ &= z, \text{ using (B4)} \end{aligned}$$

(*b*) Take $z = 1$ in (*a*) above to obtain (*b*).

(*c*) Take $z = \frac{1}{x}$ in (*a*) above to obtain (*c*).

(*d*) Since $\frac{1}{x} \cdot x = 1$, by (*c*) above, taking $\frac{1}{x}$ in place of x and x in place of y, we obtain $\frac{1}{\frac{1}{x}} = x$.

(3.6) Proposition The field axioms imply the following statements, for any $x, y, z \in \mathsf{F}$:

(a) $0x = 0$.

(b) If $x \neq 0$ and $y \neq 0$ then $xy \neq 0$.

(c) $(-x)y = -(xy) = x(-y)$,

(d) $(-x)(-y) = xy$.

Proof

(*a*) $0x+0x=(0+0)x=0x$. Hence (3.4)(*b*) implies that $0x=0$, and (*a*) holds.

(*b*) Next, assume $x\neq 0$, $y\neq 0$, but $xy=0$. Then (*a*) gives

$$1=\left(\frac{1}{y}\right)\left(\frac{1}{x}\right)xy=\left(\frac{1}{y}\right)\left(\frac{1}{x}\right)0=0,$$

a contradiction. Thus (*b*) holds.

(*c*) The first equality in (*c*) comes from

$$(-x)y+xy=\left(-x+x\right)y=0y=0,$$

combined with (3.4)(*c*); the other half of (*c*) is proved in the same way.

(*d*) Finally,

$$(-x)(-y)=-[x(-y)]=-[-(xy)]=xy$$

by (c) and (3.4)(*d*).

(3.7) Definition An **ordered field** is a field F which is also an ordered set, such that

(i) $x+y<x+z$ if $x, y, z\in \mathsf{F}$ and $y<z$,

(ii) $xy>0$ if $x, y\in \mathsf{F}$, $x>0$, and $y>0$.

If $x>0$, we call x positive; if $x<0$, x is negative.

(3.8) Example Q is an ordered field.

(3.9) Proposition The following statements are true in every ordered field.

(a) If $x>0$ then $-x<0$ and vice versa.

(b) If $x>0$ and $y<z$ then $xy<xz$.

(c) If $x<0$ and $y<z$ then $xy>xz$.

(d) If $x\neq 0$ then $x^2>0$. In particular, $1>0$.

(e) If $0<x<y$ then $0<\frac{1}{y}<\frac{1}{x}$.

Proof

(*a*) If $x > 0$ then $0 = -x + x > -x + 0,$ so that $-x < 0.$ If $x < 0$ then $0 = -x + x < -x + 0,$ so that $-x > 0.$ This proves (*a*).

(*b*) Since $z > y,$ we have $z - y > y - y = 0,$ hence $x(z - y) > 0,$ and therefore

$$xz = x(z - y) + xy > 0 + xy = xy.$$

This proves (*b*).

(*c*) By (*a*) and (*b*), and Proposition (3.6) (*c*),

$$-[x(z - y)] = (-x)(z - y) > 0,$$

so that $x(z - y) < 0,$ hence $xz < xy.$

(*d*) If $x > 0$, Part (*ii*) of Definition (3.7) gives $x^2 > 0.$ If $x < 0$, then $-x > 0$, hence, again by Part (*ii*) of Definition (3.7), $(-x)^2 > 0.$ But $x^2 = (-x)^2 > 0,$ by Proposition (3.6)(*d*). Hence $x^2 > 0.$ Since $1 = 1^2,$ $1 > 0.$

(*e*) Note that if $y > 0$ and $v \le 0,$ then $yv \le 0.$ But $y \cdot \left(\frac{1}{y}\right) = 1 > 0.$ Hence $\frac{1}{y} > 0.$ **(**For, otherwise, $\frac{1}{y} < 0$ forces us to say that $y \cdot \left(\frac{1}{y}\right) < 0,$ by the note, which is not the case here**)**. Likewise, $\frac{1}{x} > 0.$ If we multiply both sides of inequality $x < y$ by the positive quantity, $\left(\frac{1}{x}\right)\left(\frac{1}{y}\right),$ we obtain $\frac{1}{y} < \frac{1}{x}.$

4 THE REAL FIELD

(4.1) Theorem There exists an ordered field $\mathbb{R}$ which has the least upper bound property. Moreover, $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield.

Proof The second statement means that $\mathbb{Q}$ is a subset of $\mathbb{R}$ and that the operations of addition and multiplication in $\mathbb{R}$, when applied to members of $\mathbb{Q}$, coincide with the usual operations on rational numbers; also, the positive rational numbers are positive elements of

. The members of are called **real numbers.** Now the rest of the proof is done by construction from . We shall divide the construction into several steps.

Step 1 The members of will be a certain subsets of , called **cuts.** A cut is, by definition, any set $\alpha \subset$ with the following three properties:

(I) α is not empty and $\alpha \neq$.

(II) If $p \in \alpha$, $q \in$, and $q < p$, then $q \in \alpha$.

(III) If $p \in \alpha$, then $p < r$ for some $r \in \alpha$.

Notation The letters $p, q, r, \cdots$ will always denote rational numbers, and $\alpha, \beta, \gamma, \cdots$ will denote cuts.

Note that (III) simply says that α has no largest member; (II) implies two facts which will be used freely:

If $p \in \alpha$ and $q \notin \alpha$ then $p < q$.

If $r \notin \alpha$ and $r < s$ then $s \notin \alpha$.

Step 2 Define $\alpha < \beta$ to mean: α is a proper subset of β.

Claim: $<$ is an order [Ref. Definition (2.1)]

(i) If $\alpha < \beta$ and $\beta < \gamma$ it is clear that $\alpha < \gamma$. (As a proper subset of a proper subset is a proper subset).

(ii) It is also clear that at most one of the three relations

$$\alpha < \beta, \quad \alpha = \beta, \quad \beta < \alpha$$

can hold for any pair α, β. To show that at least one holds, assume that the first two fail. Then α is not a subset of β. Hence there is a $p \in \alpha$ with $p \notin \beta$. If $q \in \beta$, it follows that $q < p$ (since $p \notin \beta$), hence $q \in \alpha$, by (II). Thus $\beta \subset \alpha$. Since $\beta \neq \alpha$, we conclude: $\beta < \alpha$

Thus is now an ordered set.

Step 3 The ordered set has the least upper bound property.

To prove this, let A be a nonempty subset of $\mathbb{R}$, and assume that $\beta \in \mathbb{R}$ is an upper bound of A. Define γ to be the union of all $\alpha \in A$. In other words, $p \in \gamma$ if and only if $p \in \alpha$ for some $\alpha \in A$.

Claim: $\gamma \in \mathbb{R}$ and $\gamma = \sup A$.

Since A is not empty, there exists an $\alpha_0 \in A$. This α_0 is not empty. Since $\alpha_0 \subset \gamma$, $\gamma \neq \Phi$. Next $\gamma \subset \beta$ (since $\alpha \subset \beta$, $\forall \alpha \in A$), and therefore $\gamma \neq \mathbb{R}$. Thus γ satisfies property (I). To prove (II) and (III), pick $p \in \gamma$. Then $p \in \alpha_1$ for some $\alpha_1 \in A$. If $q < p$, then $q \in \alpha_1$, hence $q \in \gamma$; this proves (II). If $\gamma \in \alpha_1$ is so chosen that $r > p$, we see that $r \in \gamma$ (since $\alpha_1 \subset \gamma$), and therefore γ satisfies (III).

Thus $\gamma \in \mathbb{R}$.

It is clear that $\alpha \leq \gamma \ \forall \alpha \in A$.

Suppose $\delta < \gamma$. Then there is an $s \in \gamma$ and that $s \notin \delta$. Since $s \in \gamma$, $s \in \alpha$ for some $\alpha \in A$. Hence $\delta < \alpha$, and δ is not an upper bound of A.

This gives the desired result: $\gamma = \sup A$.

Step 4 If $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ we <u>define</u> $\alpha + \beta = \{r + s : r \in \alpha, s \in \beta\}$.

We define 0^* to be the set of all negative rational numbers. It is clear that 0^* is a cut.

Verification of the fact that the axioms for addition [see Definition (3.1)] hold in $\mathbb{R}$, *with* 0^* *playing the role of 0:*

(A1) We have to show that $\alpha + \beta$ is a cut. It is clear that $\alpha + \beta$ is a nonempty subset of $\mathbb{R}$. Take $r' \notin \alpha$, $s' \notin \beta$. Then $r' + s' > r + s$ for all choices of $r \in \alpha$, $s \in \beta$. Thus $r' + s' \notin \alpha + \beta$. It follows that $\alpha + \beta$ has property (I).

Pick $p \in \alpha + \beta$. Then $p = r + s$, with $r \in \alpha$, $s \in \beta$. If $q < p$, then $q - s < r$, so $q - s \in \alpha$, and $q = (q - s) + s \in \alpha + \beta$. Thus (II) holds. Choose $t \in \alpha$ so that $t > r$. Then $p < t + s$ and $t + s \in \alpha + \beta$. Thus (III) holds.

(A2) $\alpha + \beta = \{r + s : r \in \alpha, s \in \beta\}$. By the same definition, $\beta + \alpha = \{s + r : r \in \alpha, s \in \beta\}$. Since $r + s = s + r \ \forall r \in \mathbb{R}$, $s \in \mathbb{R}$, we have $\alpha + \beta = \beta + \alpha$.

(A3) As above, this follows from the associative law in $\mathbb{R}$.

(A4) If $r \in \alpha$ and $s \in 0^*$, then $r+s<r$, hence $r+s \in \alpha$. Thus $\alpha + 0^* \subset \alpha$. To obtain the opposite inclusion, pick $p \in \alpha$, and pick $r \in \alpha, r > p$. Then $p-r \in 0^*$, and $p = r+(p-r) \in \alpha + 0^*$. Thus $\alpha \subset \alpha + 0^*$. We conclude that $\alpha + 0^* = \alpha$.

(A5) Fix $\alpha \in$. Let β be the set of all p with the following property:

There exists $r > 0$ such that $-p-r \notin \alpha$.

In other words, some rational number smaller than $-p$ fails to be in α.

Claim: $\beta \in$ and $\alpha + \beta = 0^*$.

If $s \notin \alpha$ and $p = -s-1$, then $-p-1 \notin \alpha$, hence $p \in \beta$. So β is not empty. If $q \in \alpha$, then $-q \notin \beta$. So $\beta \neq$. Hence β satisfies (I).

Pick $p \in \beta$, and pick $r > 0$, so that $-p-r \notin \alpha$. If $q < p$, then $-q-r > -p-r$, hence $-q-r \notin \alpha$. Thus $q \in \beta$, and (II) holds. Put $t = p + \frac{r}{2}$. Then $t > p$, and $-t - \frac{r}{2} = -p-r \notin \alpha$, so that $t \in \beta$. Hence β satisfies (III). That is, we have proved that $\beta \in$.

If $\gamma \in \alpha$ and $s \in \beta$, then $-s \notin \alpha$, hence $r < -s$, $r+s<0$. Thus $\alpha + \beta \subset 0^*$.

To prove the opposite inclusion, pick $v \in 0^*$, put $w = -\frac{v}{2}$. Then $w>0$, and there is an integer n such that $nw \in \alpha$ but $(n+1)w \notin \alpha$. (Note that this depends on the fact that has the Archimedean property). Put $p = -(n+2)w$. Then $p \in \beta$, since $-p-w \notin \alpha$, and

$$v = nw + p \in \alpha + \beta.$$

Thus $0^* \subset \alpha + \beta$.

We conclude that $\alpha + \beta = 0^*$.

This β will of course be denoted by $-\alpha$.

Step 5 Having proved that the addition defined in Step 4 satisfies Axioms (A) of Definition (3.1), it follows that Proposition (3.4) is valid in , and we can prove one of the requirements of Definition (3.7):

If $\alpha, \beta, \gamma \in$ and $\beta < \gamma$, then $\alpha + \beta < \alpha + \gamma$.

Indeed, it is obvious from the definition of + in $\mathbb{Q}$ that $\alpha+\beta \subset \alpha+\gamma$; if we had $\alpha+\beta=\alpha+\gamma$, the cancellation law [Proposition (3.4)] would imply $\beta=\gamma$.

It also follows that $\alpha > 0^*$ if and only if $-\alpha < 0^*$.

Step 6 Multiplication is a little more bothersome than addition in the present context, since products of negative rationals are positive. For this reason we confine ourselves first to $\mathbb{Q}^+ = \{\alpha \in \mathbb{Q} : \alpha > 0^*\}$.

If $\alpha \in \mathbb{Q}^+$ and $\beta \in \mathbb{Q}^+$, we define

$$\alpha\beta = \{p : p \le rs \text{ for some choice of } r \in \alpha,\ s \in \beta,\ r > 0,\ s > 0\}$$

We define $1^* = \{q \in \mathbb{Q} : q < 1\}$.

Then the axioms (M) and (D) of Definition (3.1) hold, with $\mathbb{Q}^+$ in place of F and with 1^* in place of 1:

The proofs are so similar to the ones given in detail in Step 4 that we omit them.

Note, in particular, that the second requirement of Definition (3.7) holds: If $\alpha > 0^*$ and $\beta > 0^*$ then $\alpha\beta > 0^*$.

Step 7 We complete the definition of multiplication by setting $\alpha 0^* = 0^*\alpha = 0^*$, and by setting

$$\alpha\beta = \begin{cases} (-\alpha)(-\beta) & \text{if } \alpha < 0^*,\ \beta < 0^* \\ -[(-\alpha)\beta] & \text{if } \alpha < 0^*,\ \beta > 0^* \\ -[\alpha(-\beta)] & \text{if } \alpha > 0^*,\ \beta < 0^*. \end{cases}$$

The products on the right were defined in Step 6.

Having proved (in Step 6) that the axioms (M) hold in $\mathbb{Q}^+$, it is now perfectly simple to prove them in $\mathbb{Q}$, by repeated application of the identity $\gamma = -(-\gamma)$ which is a part of Proposition (3.4) (See Step 5).

The proof of the distributive law

$$\alpha(\beta+\gamma) = \alpha\beta + \alpha\gamma$$

breaks into cases. For instance, suppose $\alpha > 0^*$, $\beta < 0^*$, $\beta + \gamma > 0^*$. Then $\gamma = (\beta + \gamma) + (-\beta)$, and (since we already know that the distributive law holds in $\mathbb{Q}^+$)

$$\alpha\gamma = \alpha(\beta + \gamma) + \alpha \cdot (-\beta).$$

But $\alpha \cdot (-\beta) = -\alpha\beta$. Thus

$$\alpha\beta + \alpha\gamma = \alpha(\beta + \gamma).$$

The other cases are handled in the same way.

We have now completed the proof that $\mathbb{R}$ is an ordered field with the least upper bound property.

Step 8 We associate with each $r \in \mathbb{Q}$, the set $r^* = \{p \in \mathbb{Q} : p < r\}$. Clearly r^* is a cut; that is, $r^* \in \mathbb{R}$. These cuts satisfy the following relations:

(a) $r^* + s^* = (r + s)^*$,

(b) $r^* s^* = (rs)^*$,

(c) $r^* < s^*$ if and only if $r < s$.

To prove (*a*), choose $p \in r^* + s^*$. Then $p = u + v$, where $u < r$, $v < s$. Hence $p < r + s$, which says that $p \in (r + s)^*$.

Conversely, suppose $p \in (r + s)^*$. Then $p < r + s$. Choose t so that $2t = r + s - p$, put

$$r' = r - t, \; s' = s - t.$$

Then $r' \in r^*$, $s' \in s^*$, and $p = r' + s'$, so that $p \in r^* + s^*$. This proves (*a*). The proof of (*b*) is similar.

If $r < s$, then $r \in s^*$, but $r \notin r^*$; hence $r^* < s^*$.

If $r^* < s^*$, then there is a $p \in s^*$ such that $p \notin r^*$. Hence $r \le p < s$, so that $r < s$. This proves (*c*).

Step 9 Claim: $\mathbb{Q}$ is a subfield of $\mathbb{R}$.

We saw in Step 8 that the replacement of the rational numbers r by the corresponding "rational cuts" $r^* \in \mathbb{R}$ preserves sums, products, and order. This fact may be expressed by saying that the ordered field $\mathbb{Q}$ is *isomorphic* to the ordered field $\mathbb{Q}^*$ whose elements are rational cuts. Of course, r^* by no means the same as r, but the properties we are concerened with (arithmetic and order) are the same in the two fields.

It is the above identification of $\mathbb{Q}$ with $\mathbb{Q}^*$ which allows us to regard $\mathbb{Q}$ as a subfield of $\mathbb{R}$.

(4.2) Theorem

(a) (***Archimedean property of*** $\mathbb{R}$.) If $x \in \mathbb{R}$, $y \in \mathbb{R}$, and $x > 0$, then there is a positive integer n such that

$$nx > y.$$

(b) (***Density of*** $\mathbb{Q}$ ***in*** $\mathbb{R}$.) If $x \in \mathbb{R}$, $y \in \mathbb{R}$, and $x < y$, then there exists a $p \in \mathbb{Q}$ such that $x < p < y$. i.e., between any two real numbers there is a rational one.

Proof

(*a*) Let $A = \{nx : n \in \mathbb{N}\}$. If (*a*) were false, then y would be an upper bound of A. But then A has a least upper bound in $\mathbb{R}$. Put $\alpha = \sup A$. Since $x > 0$, $\alpha - x < \alpha$, and $\alpha - x$ is not an upper bound of A. Hence $\alpha - x < mx$ for some $m \in \mathbb{N}$. But then $\alpha < (m+1)x \in A$, which is impossible, since α is an upper bound of A.

(*b*) Since $x < y$, we have $y - x > 0$, and (*a*) furnishes a positive integer n such that

$$n(y - x) > 1.$$

Apply (*a*) again, to obtain positive integers m_1 and m_2 such that $m_1 > nx$, $m_2 > -nx$. Then

$$-m_2 < nx < m_1.$$

Hence there is an integer m (with $-m_2 \le m \le m_1$) such that

$$m - 1 \le nx < m.$$

If we combine these inequalities, we obtain

$$nx < m \le 1 + nx < ny.$$

Since $n > 0,$ it follows that

$$x < \frac{m}{n} < y.$$

This proves (*b*) with $p = \frac{m}{n}$.

(4.3) Theorem (***Existence of nth roots of positive reals***) For every real $x > 0$ and every integer $n > 0$ there is one and only one real y such that $y^n = x$ and $y > 0.$ (UQ 2006)

Remarks

- The number y above is written $\sqrt[n]{x}$ or $x^{1/n}$.
- In particular, $\sqrt{2}$ is real number (Taking $x = 2$ and $n = 2$)

Proof of the theorem That there is at most one such y is clear, since $0 < y_1 < y_2$ implies $y_1^n < y_2^n$.

Let $$E = \{t \in \square : t > 0, t^n < x\}.$$

If $t = \frac{x}{1+x}$ then $0 < t < 1.$ Hence $t^n < t < x.$ Thus $t \in E,$ and E is not empty.

If $t > 1 + x$ then $t^n > t > x,$ so that $t \notin E.$ Thus $1 + x$ is an upper bound of $E.$

Hence Theorem (4.1) implies the existence of

$$y = \sup E.$$

To prove that $y^n = x$ we will show that each of the inequalities $y^n < x$ and $y^n > x$ leads to a contradiction.

The identity $b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + \cdots + a^{n-1})$ yields the inequality

$$b^n - a^n < (b-a)nb^{n-1}$$

when $0 < a < b.$

Assume $y^n < x.$ Choose h so that $0 < h < 1$ and

$$h < \frac{x - y^n}{n(y+1)^{n-1}}.$$

Put $a = y,\ b = y + h.$ Then

$$(y+h)^n - y^n < hn(y+h)^{n-1} < hn(y+1)^{n-1} < x - y^n.$$

Thus $(y+h)^n < x,$ and $y + h \in E.$ Since $y + h > y,$ this contradicts the fact that y is an upper bound of $E.$

Assume $y^n > x.$ Put

$$k = \frac{y^n - x}{ny^{n-1}}.$$

Then $0 < k < y.$ If $t \geq y - k,$ we conclude that

$$y^n - t^n \leq y^n - (y-k)^n < kny^{n-1} = y^n - x.$$

Thus $t^n > x,$ and $t \notin E.$ It follows that $y - k$ is an upper bound of $E.$

But $y - k < y,$ which contradicts the fact hat y is the least upper bound of $E.$

Hence $y^n = x,$ and the proof is complete.

(4.4) Corollary If a and b are positive real numbers and n is a positive integer, then

$$(ab)^{1/n} = a^{1/n} b^{1/n}.$$

Proof Put $\alpha = a^{1/n},\ \beta = b^{1/n}.$ Then

$$ab = \alpha^n \beta^n = (\alpha\beta)^n,$$

since multiplication is commutative [Axiom (M2) in Definition (3.1)]. The uniqueness assertion of Theorem (4.3) shows therefore that

$$(ab)^{1/n} = \alpha\beta = a^{1/n} b^{1/n}.$$

(4.5) The Relation between Real numbers and Decimals

Let $x > 0$ be real. Let n_0 be the largest integer such that $n_0 \le x$. (Note that the existence of n_0 depends on the Archimedean property of $\mathbb{R}$.) Having chosen n_0, $n_1, \ldots, n_{k-1}$, let n_k be the largest integer such that

$$n_0 + \frac{n_1}{10} + \cdots + \frac{n_k}{10^k} \le x.$$

Let E be the set of these numbers

$$n_0 + \frac{n_1}{10} + \cdots + \frac{n_k}{10^k} \qquad (k = 0,1,2,\ldots) \tag{1}$$

Then $x = \sup E$. The decimal expansion of x is

$$n_0 \cdot n_1 n_2 n_3 \cdots n_k \cdots. \tag{2}$$

Conversely, for any infinite decimal (2) the set E of numbers (1) is bounded above, and (2) is the decimal expansion of $\sup E$.

5 THE EXTENDED REAL NUMBER SYSTEM

(5.1) Definition The **extended real number system** consists of the real field $\mathbb{R}$ and two symbols, $+\infty$ and $-\infty$. We preserve the original order in $\mathbb{R}$, and define

$$-\infty < x < +\infty \qquad \forall x \in \mathbb{R}.$$

- It is then clear that $+\infty$ is an upper bound of every subset of the extended real number system, and that every nonempty subset has a least upper bound. If, for example, E is a nonempty set of real numbers which is not bounded above in $\mathbb{R}$, then $\sup E = +\infty$ in the extended real number system.

- Similarly, $-\infty$ is a lower bound of every subset of the extended real number system, and that every nonempty subset has a greatest lower bound. If, for example, E is a nonempty set of real numbers which is not bounded below in $\mathbb{R}$, then $\inf E = -\infty$ in the extended real number system.

- The extended real number system does not form a field, but it is customary to make the following conventions:

  (a) If $x \in \mathbb{R}$ then

  $$x + \infty = +\infty,\ x - \infty = -\infty,\ \frac{x}{+\infty} = \frac{x}{-\infty} = 0.$$

  (b) If $x > 0$ then $x \cdot (+\infty) = +\infty,\ x \cdot (-\infty) = -\infty.$

  (c) If $x < 0$ then $x \cdot (+\infty) = -\infty,\ x \cdot (-\infty) = +\infty.$

- When it is desired to make the distinction between real numbers on the one hand and the symbols $+\infty$ and $-\infty$ on the other quite explicit, the former are called *finite*.

6 THE COMPLEX FIELD

(6.1) Definition A **complex number** is an ordered pair (a, b) of real numbers. We denote the set of complex numbers by $\mathbb{C}$.

- Ordered means that (a, b) and (b, a) are regarded as distinct if $a \neq b$.
- Let $x = (a, b),\ y = (c, d)$ be two complex numbers. We write $x = y$ if and only if $a = c$ and $b = d$.

(6.2) Theorem The addition and multiplication in the set of complex numbers defined by

$$x + y = (a + c, b + d).$$

$$xy = (ac - bd, ad + bc)$$

makes it into a field, with $(0, 0)$ and $(1, 0)$ in the role of 0 and 1.

Proof We simply verify the field axioms, as listed in Definition (3.1).

Let $x = (a, b),\ y = (c, d),\ z = (e, f)$.

(A1) is clear.

(A2) $x+y=(a+c,b+d)$, by the definition of addition in $\mathbb{C}$

$=(c+a,d+b)$, by the commutativity of addition in $\mathbb{R}$.

$=y+x.$

(A3) $(x+y)+z=(a+c,b+d)+(e,f)$

$=(a+c+e,b+d+f)$, by the associativity of addition in $\mathbb{R}$.

$=(a,b)+(c+e,d+f)$

$=x+(y+z).$

(A4) $x+0=(a,b)+(0,0)=(a,b)=x.$

(A5) Put $-x=(-a,-b).$ Then

$x+(-x)=(a,b)+(-a,-b)=(0,0)=0.$

(M1) is clear.

(M2) $xy=(ac-bd,ad+bc)$, by the definition of multiplication in $\mathbb{C}$

$=(ca-db,da+cb)$, by the commutativity of multiplication in $\mathbb{R}$.

$=yx.$

(M3) $(xy)z=(ac-bd+ad+bc)(e,f)$

$=(ace-bde-adf-bcf,\ acf-bdf+ade+bce)$

$=(a,b)(ce-df,\ cf+de)$

$=x(yz).$

(M4) $1x=(1,0)(a,b)=(a,b)=x.$

(M5) If $x\neq 0$ then $(a,b)\neq(0,0),$ which means that at least one of the real numbers a, b is different from 0. Hence $a^2+b^2>0,$ by Proposition (3.9)(d), and we can define

$$\frac{1}{x}=\left(\frac{a}{a^2+b^2},\frac{-b}{a^2+b^2}\right).$$

Then $$x \cdot \frac{1}{x} = (a, b)\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right) = (1, 0) = 1.$$

(D) $$x(y+z) = (a, b)(c+e, d+f)$$

$$= (ac+ae-bd-bf,\ ad+af+bc+be)$$

$$= (ac-bd,\ ad+bc)+(ae-bf,\ af+be)$$

$$= xy + xz.$$

THE REAL FIELD AS A SUBFIELD OF THE COMPLEX FIELD.

(6.3) Theorem For any real numbers a and b, we have

$$(a, 0)+(b, 0) = (a+b, 0),\ (a, 0)(b, 0) = (ab, 0).$$

The proof of the above Theorem is trivial. Theorem (6.3) shows that the complex numbers of the form $(a, 0)$ have the same arithmetic properties as the corresponding real numbers a. We can therefore identify $(a, 0)$ with a. This identification gives us the real field as a subfield of the complex field.

(6.4) Definition $i = (0, 1)$.

(6.5) Theorem $i^2 = -1$.

Proof $i^2 = (0, 1)(0, 1) = (-1, 0) = -(1, 0) = -1.$

(6.6) Theorem If $a, b \in \mathbb{R}$, then $(a, b) = a + bi$.

Proof $$a + bi = (a, 0)+(b, 0)(0, 1)$$

$$= (a, 0)+(0, b)$$

$$= (a, b).$$

(6.7) Definition If $a, b \in \mathbb{R}$, and $z = a + bi$, then the complex number $\bar{z} = a - bi$ is called the **conjugate** of z. The numbers a and b are the **real part** and the **imaginary part** of z, respectively.

We shall occasionally write $a = \mathrm{Re}(z),\quad b = \mathrm{Im}(z).$

(6.8) Theorem If z and w are complex, then

(a) $\overline{z+w} = \overline{z} + \overline{w}.$

(b) $\overline{zw} = \overline{z} \cdot \overline{w}.$

(c) $z + \overline{z} = 2\,\mathrm{Re}(z),\ z - \overline{z} = 2i\,\mathrm{Im}(z).$

(d) $z\overline{z}$ is real and positive (except when $z = 0$).

Proof (*a*), (*b*) and (*c*) are quite trivial. To prove (*d*), write $z = a + bi,$ and note that $z\overline{z} = a^2 + b^2.$

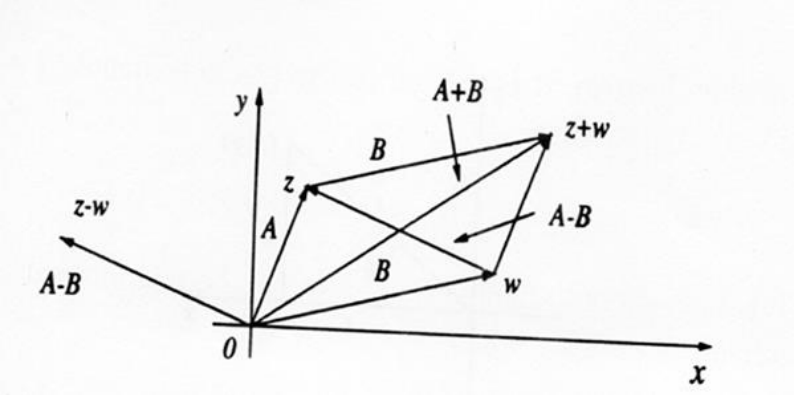


Figure: *Vector representation of complex numbers (addition and subtraction)*

(7.1) Definition If z is a complex number, its **absoloute value** $|z|$ is the nonnegative square root of $z\overline{z}$; i.e., $|z| = (z\overline{z})^{1/2}$.

(7.2) Remarks

- The existence (and uniqueness) of $|z|$ follows from Theorem (4.3) and part (d) of Theorem (6.8).
- When x is **real**, then $\overline{x} = x,$ hence $|x| = \sqrt{x^2}.$ Thus $|x| = \begin{cases} x, \text{ if } x \geq 0 \\ -x, \text{ if } x < 0 \end{cases}$

(7.3) Theorem Let z and w be complex numbers, then

(a) $|z| > 0$ unless $z = 0,$ $|0| = 0.$

(b) $|\overline{z}| = |z|.$ unless $z = 0,$ $|0| = 0.$

(c) $|zw| = |z||w|$.

(d) $|\mathrm{Re}\, z| \le |z|$.

(e) $|z + w| = |z| + |w|$.

Proof (*a*) and (*b*) are trivial.

(*c*) Put $z = a + bi,\ w = c + di,$ with $a, b, c, d \in \mathbb{R}$. Then

$$|zw|^2 = (ac - bd)^2 + (ad + bc)^2 = (a^2 + b^2)(c^2 + d^2) = |z|^2 |w|^2.$$

From the above, we have $|zw|^2 = \big(|z||w|\big)^2$. Now (*c*) follows from the uniqueness assertion of Theorem (4.3).

(*d*) Note that $a^2 \le a^2 + b^2,$ hence

$$|a| = \sqrt{a^2} \le \sqrt{a^2 + b^2}.$$

i.e., $$|\mathrm{Re}\, z| \le |z|.$$

(*e*) Note that $\bar{z}w$ is the conjugate of $z\bar{w},$ so that $z\bar{w} + z\bar{w} = 2\,\mathrm{Re}(z\bar{w})$. Hence

$$|z + w|^2 = (z + w)(\bar{z} + \bar{w}) = z\bar{z} + z\bar{w} + \bar{z}w + w\bar{w}$$

$$= |z|^2 + 2\,\mathrm{Re}(z\bar{w}) + |w|^2$$

$$\le |z|^2 + 2|z\bar{w}| + |w|^2, \text{ using } (d) \text{ above}$$

$$= |z|^2 + 2|z||w| + |w|^2, \text{ since } |\bar{w}| = |w|$$

$$= \big(|z| + |w|\big)^2.$$

Now (*e*) follows by taking square roots.

(7.4) Notation If $x_1, \ldots, x_n$ are complex numbers, we write

$$x_1 + \ldots + x_n = \sum_{j=1}^{n} x_j.$$

(7.5) Theorem (Schwarz inequality) If $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are complex numbers, then

$$\left|\sum_{j=1}^{n} a_j \overline{b_j}\right|^2 \le \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2. \qquad \text{(UQ 2006)}$$

Proof

Put $A = \sum_{j=1}^{n} |a_j|^2$, $B = \sum_{j=1}^{n} |b_j|^2$, $C = \sum_{j=1}^{n} a_j \overline{b_j}$.

Case (*i*) If $B = 0$, then $|b_1|^2 = \ldots = |b_n|^2 = 0$, implies $b_1 = \ldots = b_n = 0$, and hence $C = \sum_{j=1}^{n} a_j \overline{b_j} = 0$, and we get the result (In this case equality occurs).

Case (*ii*) Assume $B > 0$. Then

$$\sum_{j=1}^{n} |Ba_j - Cb_j|^2 = \sum_{j=1}^{n} (Ba_j - Cb_j)\overline{(Ba_j - Cb_j)}, \text{ by the Definition (7.1)}$$

$$= B^2 \sum_{j=1}^{n} |a_j|^2 - B\bar{C} \sum_{j=1}^{n} a_j \overline{b_j} - BC \sum_{j=1}^{n} \overline{a_j} b_j + |C|^2 \sum_{j=1}^{n} |b_j|^2$$

$$= B^2 A - B\bar{C}C - BC\bar{C} + |C|^2 B$$

$$= B^2 A - B|C|^2$$

$$= B(AB - |C|^2).$$

Since each term in the L.H.S. is nonnegative, we see that $B(AB - |C|^2) \ge 0$.

Since $B > 0$, it follows that $AB - |C|^2 \ge 0$.

i.e., $$\sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2 - \left|\sum_{j=1}^{n} a_j \overline{b_j}\right|^2 \ge 0,$$ the desired inequality.

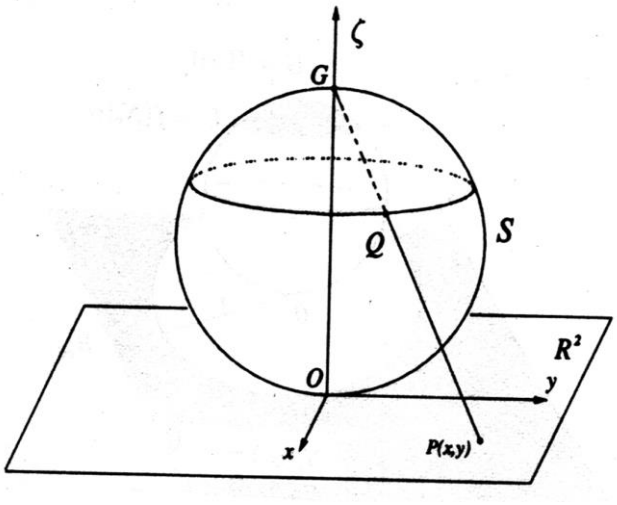
***(7.6) Spherical representation of complex number**

Let the complex plane $\mathcal{P}$ be tangent to the unit sphere $\mathcal{S}$ such that the complex number z = 0 is the *south pole* (denoted by *O*) on $\mathcal{S}$ and let *G* be the *north pole* (See figure). Then *OG*, the diameter of $\mathcal{S}$, is perpendicular to $\mathcal{P}$. Along *OG* draw a line and consider it as ζ axis. Then *G*

is on the positive ζ axis at (0, 0, 1). Corresponding to any point $P(x,y)$ on $\mathcal{P}$ the line GP intersecting $\mathcal{S}$ at *one and only one* point $P'(\xi,\eta,\zeta)$, where

$$\xi = \frac{x}{x^2+y^2+1},\ \eta = \frac{y}{x^2+y^2+1},\ \varsigma = \frac{x^2+y^2}{x^2+y^2+1}.$$

Thus we can represent any complex number in $\mathcal{P}$ by a point on the sphere $\mathcal{S}$. The point G itself corresponds to the point at infinity of the plane. The set of all points of the complex plane including the point at infinity is called **entire complex plane** or the **extended complex plane**. The above method for mapping the plane into the sphere is called *stereographic projection*. The sphere is called **Riemann sphere**.



7 EUCLIDEAN SPACES

Let k be a positive integer. $\mathbb{R}^k$ is the set of all ordered k-tuples $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ where $x_i \in \mathbb{R}$. Then $\mathbb{R}^k$ is a **vector space over** $\mathbb{R}$ under the following operations of addition and scalar multiplication: For $\mathbf{x} = (x_1, \ldots, x_k)$, $\mathbf{y} = (y_1, \ldots, y_k) \in \mathbb{R}^k$ and $\alpha \in \mathbb{R}$

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, \ldots, x_k + y_k)$$

$$\alpha\mathbf{x} = (\alpha x_1, \ldots, \alpha x_k)$$

(For details refer the study material Linear Algebra, Page 101) The zero element of $\mathbb{R}^k$ is $\mathbf{0} = (0, \ldots, 0)$, the k tuple with all its coordinates 0.

(7.7) Definitions **Inner product** (or **scalar product**) of $\mathbf{x}$ and $\mathbf{y}$ is

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{k} x_i y_i$$

and **the norm of x** is $|\mathbf{x}| = (\mathbf{x} \cdot \mathbf{x})^{1/2} = \left(\sum_{i=1}^{k} x_i^2 \right)^{1/2}$.

(7.8) Theorem Suppose $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^k$ and $\alpha \in \mathbb{R}$. Then

(a) $|\mathbf{x}| \geq 0$;

(b) $|\mathbf{x}| = 0$ if and only if $\mathbf{x} = \mathbf{0}$, the zero vector;

(c) $|\alpha\mathbf{x}| = |\alpha||\mathbf{x}|$;

(d) $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$;

(e) $|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$;

(f) $|\mathbf{x} - \mathbf{z}| \leq |\mathbf{x} - \mathbf{y}| + |\mathbf{y} - \mathbf{z}|$.

Proof

(*a*) By definition, $|\mathbf{x}| = (\mathbf{x} \cdot \mathbf{x})^{1/2} = \left(\sum_{i=1}^{k} x_i^2 \right)^{1/2}$, and hence $|\mathbf{x}| \geq 0$.

(*b*) $|\mathbf{x}| = 0$ if and only if $\left(\sum_{i=1}^{k} x_i^2 \right)^{1/2} = 0$ if and only $\sum_{i=1}^{k} x_i^2 = 0$ if and only if $x_i^2 = 0$ for $i = 1, \ldots, k$ if and only if $x_i = 0$ for $i = 1, \ldots, k$ if and only if $\mathbf{x} = (x_1, \ldots, x_k) = (0, \ldots, 0) = \mathbf{0}$.

(*c*) $|\alpha\mathbf{x}| = \left(\sum_{i=1}^{k} (\alpha x_i)^2 \right)^{1/2} = \alpha \left(\sum_{i=1}^{k} x_i^2 \right)^{1/2} = \alpha |\mathbf{x}|$.

(*d*) $|\mathbf{x} \cdot \mathbf{y}| = \left| \sum_{i=1}^{k} x_i y_i \right|$, by the definition of the norm

$\leq \left| \sum_{i=1}^{k} x_i^2 \right|^{1/2} \left| \sum_{i=1}^{k} y_i^2 \right|^{1/2}$, by Schwarz inequality

$\leq |\mathbf{x}||\mathbf{y}|$.

(*e*) $|\mathbf{x} + \mathbf{y}|^2 = (\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y})$, by the definition of the norm

$= \mathbf{x} \cdot \mathbf{x} + 2\mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{y}$

$\leq |\mathbf{x}|^2 + 2|\mathbf{x}||\mathbf{y}| + |\mathbf{y}|^2$, using (*d*) above

$\leq (|\mathbf{x}| + |\mathbf{y}|)^2$, so that (*e*) is proved.

(f) $$|\mathbf{x}-\mathbf{z}| = \Big|\underset{\mathbf{a}}{\mathbf{x}-\mathbf{y}}+\underset{\mathbf{b}}{\mathbf{y}-\mathbf{z}}\Big|$$

$$\leq |\mathbf{a}|+|\mathbf{b}|, \text{ by } (e) \text{ above}$$

$$= |\mathbf{x}-\mathbf{y}|+|\mathbf{y}-\mathbf{z}|.$$

(7.9) Corollary $|-\mathbf{x}| = |\mathbf{x}|$.

Proof $$|-\mathbf{x}| = |-1||\mathbf{x}|, \text{ by taking } \alpha = -1 \text{ in } (c) \text{ of Theorem 7.2}$$

$$= 1|\mathbf{x}|$$

$$= |\mathbf{x}|.$$

(7.10) Example Suppose $\mathbf{a} \in \mathbb{R}^k$, $\mathbf{b} \in \mathbb{R}^k$. Find $\mathbf{c} \in \mathbb{R}^k$ and $r > 0$ such that $|\mathbf{x}-\mathbf{a}| = 2|\mathbf{x}-\mathbf{b}|$ if and only if $|\mathbf{x}-\mathbf{c}| = r$. (UQ 2006)

Chapter 2

BASIC TOPOLOGY - METRIC SPACES

(1.1) Definition A set X, whose elements we shall call *points*, is said to be a **metric space** if with any two points p and q of X there is associated a real number $d(p,q)$, called the **distance** from p to q, such that

(a) $d(p,q) > 0$ if $p \neq q$; $d(p,p) = 0$;
(b) $d(p,q) = d(q,p)$;
(c) $d(p,q) \leq d(p,r) + d(r,q)$, for any $r \in X$.

Any function $d : X \times X \to \mathbb{R}$ with the above properties is called a **distance function** or a **metric.**

(1.2) Remark Every subset Y of a metric space X is a metric space in its own right, with the same distance function. For it is clear that if conditions (*a*) to (*c*) of Definition (1.1) hold for $p,q,r \in X$, they also hold if we restrict p, q, r to lie in Y.

(1.3) Example

In the Euclidean space $\mathbb{R}^k$, d defined by

$$d(\mathbf{x},\mathbf{y}) = |\mathbf{x}-\mathbf{y}| \qquad \left(\mathbf{x},\ \mathbf{y} \in \mathbb{R}^{\mathbf{k}}\right)$$

is a distance function.

Verification: Here $X = \mathbb{R}^k$.

(*a*) $d(\mathbf{x},\mathbf{y}) = |\mathbf{x}-\mathbf{y}| > 0$ if $\mathbf{x} \neq \mathbf{y}$, by (*b*) of Theorem 7.2 of Chapter 1.

(*b*)
$$\begin{aligned} d(\mathbf{x},\mathbf{y}) &= |\mathbf{x}-\mathbf{y}| \\ &= |-(\mathbf{y}-\mathbf{x})| \\ &= |\mathbf{y}-\mathbf{x}|, \text{ by Corollary 7.3 in Chapter 1} \\ &= d(\mathbf{y},\mathbf{x}). \end{aligned}$$

(*b*)
$$\begin{aligned} d(\mathbf{x},\mathbf{y}) &= |\mathbf{x}-\mathbf{y}| \\ &\leq |\mathbf{x}-\mathbf{y}| + |\mathbf{y}-\mathbf{z}|, \text{ by (}f\text{) of Theorem 7.2 Chapter 1} \end{aligned}$$

$$= d(\mathbf{x},\mathbf{z}) + d(\mathbf{z},\mathbf{y})$$

In particular, $\mathbb{R}^1$ (the real line) and $\mathbb{R}^2$ (the plane or the complex plane) are metric spaces. Also note that in $\mathbb{R}^1$ and $\mathbb{R}^2$ the norm is just the absolute value of the corresponding real or complex number.

(1.4) Remark Every subset of a Euclidean space is a metric space, by Remark (1.2).

(1.5) Definitions

- By the **segment** (a,b) we mean the set of all real numbers x such that $a < x < b$.

  i.e., $(a,b) = \{x \in \mathbb{R} : a < x < b\}$.

- By the **interval** $[a,b]$ we mean the set of all real numbers x such that $a \le x \le b$.

  i.e., $[a,b] = \{x \in \mathbb{R} : a \le x \le b\}$.

- The **half open interval** $[a,b)$ is $[a,b) = \{x \in \mathbb{R} : a \le x < b\}$.

- The **half open interval** $(a,b]$ is $(a,b] = \{x \in \mathbb{R} : a < x \le b\}$.

- If $a_i < b_i$ for $i = 1, \ldots, k$, the set of all points $\mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k$ whose coordinates satisfy the inequalities $a_i \le x_i \le b_i$ $(1 \le i \le k)$ is called a ***k*-cell.** In particular, a 1-cell is an interval, a 2-cell is a rectangle, etc.

- If $\mathbf{x} \in \mathbb{R}^k$ and $r > 0$, the **open ball** B with center at $\mathbf{x}$ and radius r is defined to the set of all $\mathbf{y} \in \mathbb{R}^k$ such that $|\mathbf{y} - \mathbf{x}| < r$.

- If $\mathbf{x} \in \mathbb{R}^k$ and $r > 0$, the **closed ball** B with center at $\mathbf{x}$ and radius r is defined to the set of all $\mathbf{y} \in \mathbb{R}^k$ such that $|\mathbf{y} - \mathbf{x}| \le r$.

- A set $E \subset \mathbb{R}^k$ is **convex** if

  $$\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in E$$

  whenever $\mathbf{x} \in E$, $\mathbf{y} \in E$, and $0 < \lambda < 1$.

(1.6) Proposition Balls are convex.

Proof Let B be an open ball with center at $\mathbf{x}$ and radius r. Let $\mathbf{y} \in B$, $\mathbf{z} \in B$, and $0 < \lambda < 1$. We have to show that $\lambda\mathbf{y} + (1-\lambda)\mathbf{z} \in B$; for this we have to show that $|\lambda\mathbf{y} + (1-\lambda)\mathbf{z} - \mathbf{x}| < r$.

Now $\mathbf{y} \in B$, $\mathbf{z} \in B$ implies $|\mathbf{y} - \mathbf{x}| < r$ and $|\mathbf{z} - \mathbf{x}| < r$. Also

$$\begin{aligned} |\lambda\mathbf{y} + (1-\lambda)\mathbf{z} - \mathbf{x}| &= |\lambda(\mathbf{y} - \mathbf{x}) + (1-\lambda)(\mathbf{z} - \mathbf{x})| \\ &\leq \lambda|\mathbf{y} - \mathbf{x}| + (1-\lambda)|\mathbf{z} - \mathbf{x}| \\ &\leq \lambda r + (1-\lambda)r \\ &= r, \end{aligned}$$

as desired.

The same proof applies to closed balls with < changed to $\leq$.

(1.7) Proposition *k*- cells are convex.

(1.8) Definitions Let X be a metric space. All points and sets mentioned below are understood to be elements and subsets of X:

- A **neighborhood** of a point p with **radius** r is the set

$$N_r(p) = \{q \in X : d(p,q) < r\}.$$

- A point p is a **limit point** of the set E if *every* neighborhood of p contains a point $q \neq p$ such that $q \in E$; in other words, p is a limit point of the set E if *every* neighborhood of p intersects E at a point *other than* the point p.
- If $p \in E$ and p is not a limit point of E, then p is called an **isolated point** of E.
- E is **closed** if every limit point of E is a point of E.
- A point p is an **interior point** of E if there is a neighborhood N of p such that $N \subset E$.
- E is **open** if every point of E is an interior point of E.

- The **complement** of E (denoted by E^c or $\sim E$) is the set $E^c = \{p \in X : p \notin E\}$.
- E is **perfect** if E is closed and if every point of E is a limit point of E.
- E is **bounded** if there is a real number M and a point $q \in X$ such that $d(p,q) < M$ for all $p \in E$.
- E is **dense** in X if every point of X is a limit point of E, or a point of E (or both).

(1.9) Remarks

- In $\mathbb{R}^1$ neighborhoods are segments.
- In $\mathbb{R}^2$ neighborhoods are interiors of circles.

(1.10) Theorem Every neighborhood is an open set.

Proof Consider a neighborhood $E = N_r(p)$, and let $q \in E$. Then there is a positive real number h such that

$$d(p,q) = r - h.$$

For all points s such that $d(q,s) < h$, we have then

$$d(p,s) \le d(p,q) + d(q,s) < r - h + h = r,$$

so that $s \in E$. Thus q is an interior point of E.

(1.11) Theorem If p is a limit point of a set E, then every neighborhood of p contains infinitely many points of E.

Proof Suppose there is a neighborhood N of p which contains only a finite number of points of E. Let $q_1, \ldots, q_n$ be those points of $N \cap E$, which are distinct from p, and put

$$r = \min_{1 \le m \le n} d(p,q_m) = \min\{d(p,q_m) : 1 \le m \le n\}.$$

The minimum of a finite set of positive numbers is clearly positive, so that $r > 0$.

The neighborhood $N_r(p)$ contains no point q of E such that $q \ne p$, so that p is not a limit point of E. This contradiction establishes the theorem.

(1.12) Corollary A finite point set has no limit points.

Proof Let E be a finite point set. Suppose E has a limit point, say p. The by Theorem (1.11) every neighborhood of p must contain infinitely many points of E, which is not possible because E is a finite set. Hence E has no limit point.

(1.13) Examples Let us consider the following subsets of $\mathbb{R}^2$:

(a) The set of all complex z such that $|z| < 1$.

(b) The set of all complex z such that $|z| \le 1$.

(c) A finite set.

(d) The set of all integers.

(e) Consider the set $E = \left\{ \frac{1}{n} : n = 1, 2, 3, \ldots \right\}$. Let us note that this set E has a limit point (namely, $z = 0$) but that no point of E is a limit point of E; here note the difference between having a limit point and containing one.

(f) The set of all complex numbers (that is, $\mathbb{R}^2$).

(g) The segment (a, b).

Let us note that (*d*), (*e*), (*g*) can be regarded also as subsets of $\mathbb{R}^1$.

Some properties of the above sets are tabulated below:

	Closed	*Open*	*Perfect*	*Bounded*
(*a*)	No	Yes	No	Yes
(*b*)	Yes	No	Yes	Yes
(*c*)	Yes	No	No	Yes
(*d*)	Yes	No	No	No
(*e*)	No	No	No	Yes
(*f*)	Yes	Yes	Yes	No
(*g*)	No		No	Yes

In (*g*), we left the second entry blank. The reason is that the segment (a, b) is not open if we regard it as a subset of $\mathbb{R}^2$, but it is not a subset of $\mathbb{R}^1$.

(1.14) Theorem Let $\{E_\alpha\}$ be a (finite or infinite) collection of sets E_α. Then

$$\left(\bigcup_\alpha E_\alpha \right)^C = \bigcap_\alpha \left(E_\alpha^C \right).$$

Proof If $x \in \left(\bigcup_\alpha E_\alpha \right)^C$, then $x \notin \bigcup_\alpha E_\alpha$, hence $x \notin E_\alpha \ \forall \alpha$, hence $x \in E_\alpha^C \ \forall \alpha$, so that $x \in \bigcap E_\alpha^C$. Thus $\left(\bigcup_\alpha E_\alpha \right)^C \subset \bigcap_\alpha \left(E_\alpha^C \right)$.

Conversely, if $x \in \bigcap_\alpha \left(E_\alpha^C \right)$, then $x \in E_\alpha^C \ \forall \alpha$, hence $x \notin E_\alpha \ \forall \alpha$, hence $x \notin \bigcup_\alpha E_\alpha$, so that $x \in \left(\bigcup_\alpha E_\alpha \right)^C$. Thus $\bigcap_\alpha \left(E_\alpha^C \right) \subset \left(\bigcup_\alpha E_\alpha \right)^C$.

Hence $$\left(\bigcup_\alpha E_\alpha \right)^C = \bigcap_\alpha \left(E_\alpha^C \right).$$

(1.15) Theorem A set E is open if and only if its complement is closed.

Proof First, suppose E^C is closed. Choose $x \in E$. Then $x \notin E^C$, and x is not a limit point of E^C. Hence there exists a neighborhood N of x such that $E^C \cap N = \Phi$, i.e., $N \subset E$. Thus x is an interior point of E, and E is open.

Next, suppose E is open. Let x be a limit point of E^C. Then every neighborhood of x contains a point of E^C, so that x is not an interior point of E. Since E is open, this means that $x \in E^C$. It follows that E^C is closed.

(1.16) Corollary A set F is closed if and only if its complement is open.

(1.17) Theorem

(a) For any collection $\{G_\alpha\}$ of open sets, $\bigcup_\alpha G_\alpha$ is open.

(b) For any collection $\{F_\alpha\}$ of open sets, $\bigcap_\alpha F_\alpha$ is closed.

(c) For any finite collection $G_1, \ldots, G_n$ of open sets, $\bigcap_{i=1}^n G_i$ is open.

(d) For any finite collection $F_1, \ldots, F_n$ of closed sets, $\bigcup_{i=1}^n F_i$ is closed.

Proof

(*a*) To show that $\bigcup_{\alpha} G_\alpha$ is open, it is enough to show that each point of $\bigcup_{\alpha} G_\alpha$ is an interior point. Let $x \in \bigcup_{\alpha} G_\alpha$, then $x \in G_\alpha$ for some α. Since G_α is open , x is an interior point of G_α, hence x is an interior point of $\bigcup_{\alpha} G_\alpha$, and so $\bigcup_{\alpha} G_\alpha$ is open.

(*b*) By Theorem (1.14),

$$\left(\bigcap_{\alpha} F_\alpha \right)^C = \bigcup_{\alpha} F_\alpha^C, \tag{1}$$

and F_α^C is open, by Theorem (1.15). Hence (*a*) implies that R.H.S. of (1) is open. Hence its complement $\bigcap_{\alpha} F_\alpha$ is closed.

(*c*) Let $x \in \bigcap_{i=1}^{n} G_i$. Then $x \in G_i$ for $i = 1, \ldots, n$. Since each G_i is open, x is an interior point of each G_i and hence there exist neighborhoods N_i of x, with radii r_i, such that $x \in N_i \subset G_i$, $i = 1, \ldots, n$. Put

$$r = \min(r_1, \ldots, r_n),$$

and let N be the neighborhood of x of radius r. Then $x \in N \subset G_i$ for $i = 1, \ldots, n$, so that $x \in N \subset \bigcap_{i=1}^{n} G_i$, and hence $\bigcap_{i=1}^{n} G_i$ is open.

(*d*) By Theorem (1.14),

$$\left(\bigcup_{i=1}^{n} F_i \right)^C = \bigcap_{i=1}^{n} F_i^C, \tag{2}$$

and F_i^C is open, by Theorem (1.15). Hence (*c*) implies that R.H.S. of (2) is open. Hence its complement $\bigcup_{i=1}^{n} F_i$ is closed.

(1.18) Example (*Example to show that intersection of infinite collection of open sets need not be open*) In parts (*c*) and (*d*) of the preceding theorem, the finiteness of the collections is essential. For let $G_n = \left(-\frac{1}{n}, \frac{1}{n} \right) (n = 1, 2, 3, \ldots)$. Then G_n, being a segment in $\mathbb{R}^1$ is a

neighborhood in $\mathbb{R}^1$ and by Theorem (1.10), is an open subset of $\mathbb{R}^1$. But, $\bigcap_{n\in\mathbb{N}} G_n = \bigcap_{n\in\mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$ is a finite set and hence is <u>not</u> an open subset of $\mathbb{R}^1$.

(1.19) Example (*Example to show that union of infinite collection of closed sets need not be closed*) The set of rational numbers is the union of a countable collection of closed sets each of which contains exactly one rational number. i.e.,

$$\mathbb{Q} = \bigcup_{a\in\mathbb{Q}} \{a\}.$$

Though each $\{a\}$ is closed, $\mathbb{Q}$ is not closed (For, we can find a sequence in $\mathbb{Q}$ which converges to $\sqrt{2} \notin \mathbb{Q}$.)

(1.20) Definition If X is a metric space, if $E \subset X$, and if E' denotes the set of all limit points of E in X, then the **closure of** E is the set $\bar{E} = E \cup E'$.

(1.21) Theorem If X is a metric space and $E \subset X$, then

(a) $\bar{E}$ is closed,

(b) $E = \bar{E}$ if and only if E is closed,

(c) $\bar{E} \subset F$ for every closed set $F \subset X$ such that $E \subset F$.

By (*a*) and (*c*), $\bar{E}$ is the **smallest closed subset** of X that contains E.

Proof of the theorem

(*a*) To show that $\bar{E}$ is closed, we show that its complement $\left(\bar{E}\right)^C$ is open.

If $p \in X$ and $p \in \left(\bar{E}\right)^C$, then $p \notin \bar{E}$ implies (by means of Definition (1.20)) that p is neither a point of E nor a limit point of E. Hence p has a neighborhood which does not intersect E. This neighborhood of p will be wholly contained in $\left(\bar{E}\right)^C$, showing that p is an interior point of $\left(\bar{E}\right)^C$. Since p is an arbitrary point of $\left(\bar{E}\right)^C$, this shows that $\left(\bar{E}\right)^C$ is open. Hence its complement $\bar{E}$ is closed.

(*b*) By (*a*), $\bar{E}$ is closed, hence $E = \bar{E}$ implies that E is closed. Conversely, if E is closed, then $E' \subset E$ (by Definition of closed set in (1.8) and Definition (1.20)), hence $\bar{E} = E$.

(*c*) If F is closed and $F \supset E$, then $F \supset F'$, hence $F \supset E'$. Thus $F \supset \bar{E}$.

(1.22) Theorem Let E be a nonempty set of real numbers which is bounded above. Let $y = \sup E$. Then $y \in \bar{E}$. Hence $y \in E$ if E is closed.

Proof

Case 1) If $y \in E$ then $y \in \bar{E}$.

Case 2) Assume $y \notin E$. Since $y = \sup E$, for every $h > 0$ there exists then a point $x \in E$ such that $y - h < x < y$, for otherwise $y - h$ would be an upper bound of E. Thus y is a limit point of E. Hence $y \in \bar{E}$.

(1.23) Definitions

Suppose $E \subset Y \subset X$, where X is a metric space:

- To say that E is an **open subset** of X means that to each point $p \in E$ there is associated a positive number r such that the conditions $d(p,q) < r$, $q \in X$ imply that $q \in E$.
- E is an **open relative to** Y if to each point $p \in E$ there is associated a positive number r such that the conditions $d(p,q) < r$, $q \in Y$ imply that $q \in E$.

(1.24) Theorem Suppose $Y \subset X$. A subset E of Y is open relative to Y if and only if $E = Y \cap G$ for some open subset G of X.

Proof Suppose E is open relative to Y. To each $p \in E$ there is a positive number r_p such that the conditions $d(p,q) < r_p$, $q \in Y$ imply that $q \in E$. Let

$$V_p = \{q \in X \mid d(p,q) < r_p\}$$

and define

$$G = \bigcup_{p \in E} V_p.$$

Then G is an open subset of X, by Theorems (1.10) and (1.15).

Since $p \in V_p \ \forall p \in E$, it is clear that $E \subset G \cap Y$.

By our choice of V_p, we have $V_p \cap Y \subset E \ \forall p \in E$, so that $G \cap Y \subset E$. Thus $E = G \cap Y$, and one half of the theorem is proved.

Conversely, if G is open in X and $E = G \cap Y$, every $p \in E$ has a neighborhood $V_p \subset G$. Then $V_p \cap Y \subset E$, so that E is open relative to Y.

Chapter 3

CONTINUITY

1 Limits of Functions

(1.1) Definition Let X and Y be metric spaces; suppose $E \subset X$, f maps E into Y, and p is a limit point of E. We write $f(x) \to q$ as $x \to p$, or

$$\lim_{x \to p} f(x) = q$$

if there is a point $q \in Y$ with the following property: For every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$d_Y(f(x), q) < \varepsilon$$

for all points $x \in E$ for which

$$0 < d_X(x, p) < \delta.$$

The symbols d_X and d_Y refer to the distances in X and Y, respectively.

If X and/or Y are replaced by the real line $\mathbb{R}$, the complex plane $\mathbb{C}$, or by some Euclidean spaces $\mathbb{R}^k$, the distances d_X and d_Y are of course replaced by absolute values, or by appropriate norms.

It should be noted that $p \in X$, but that p need not be a point of E in the above *definition.*

Moreover, even if $p \in E$, *we may very well have* $f(p) \neq \lim_{x \to p} f(x)$.

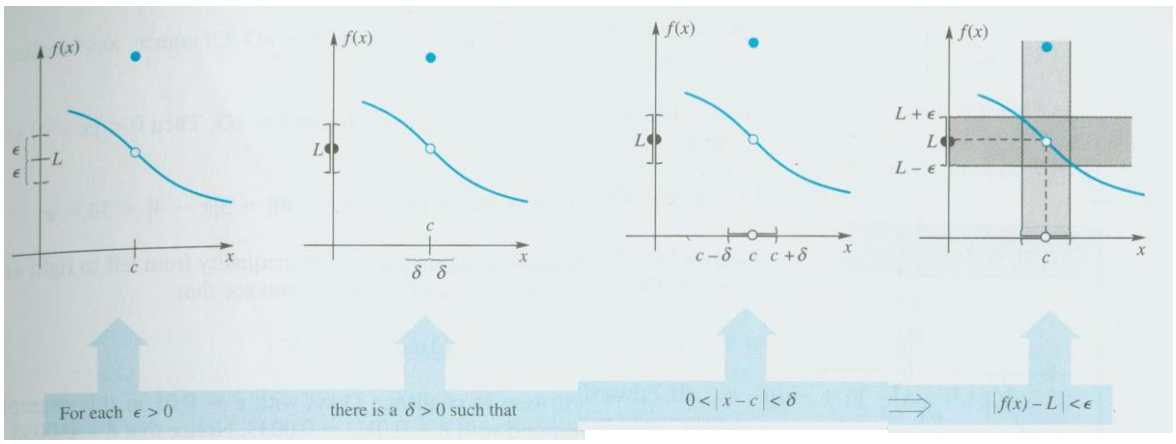


Figure: Limit at c of the given function f from $\mathbb{R}$ to $\mathbb{R}$ is L

(1.2) Theorem Let $X, Y, E, f,$ and p be as in Definition (1.1). Then

$$\lim_{x\to p} f(x) = q \tag{1}$$

if and only if

$$\lim_{n\to\infty} f(p_n) = q \tag{2}$$

for every sequence $\{p_n\}$ in E such that

$$p_n \neq p, \qquad \lim_{n\to\infty} p_n = p. \tag{3}$$

Proof

Suppose (1) holds. Choose $\{p_n\}$ in E satisfying (3). Let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that

$$d_Y(f(x), q) < \varepsilon$$

if $x \in E$ and $0 < d_X(x, p) < \delta$. Also, there exists N such that $n > N$ implies

$$0 < d_X(p_n, p) < \delta.$$

Thus for $n > N$, we have

$$d_Y(f(p_n), q) < \varepsilon,$$

which shows that (2) holds.

Conversely, suppose (1) is false. Then there exist some $\varepsilon > 0$ such that for every $\delta > 0$ there exists a point $x \in E$ (depending on δ), for which

$$d_Y(f(x), q) \geq \varepsilon$$

but $0 < d_X(x, p) < \delta$. Taking $\delta_n = \dfrac{1}{n}$ $(n = 1, 2, \ldots)$, we thus find a sequence in E satisfying (3) for which (2) is false.

(1.3) Corollary If f has a limit at p, this limit is unique.

Proof

Suppose f has a limit at p. Also, suppose that $\lim_{x\to p} f(x) = q$ and $\lim_{x\to p} f(x) = r$.

Claim $q = r$.

By Theorem (1.2), $\lim_{x\to p} f(x) = q$ implies that

$$\lim_{n\to\infty} f(p_n) = q \tag{4}$$

for every sequence $\{p_n\}$ in E such that

$$p_n \neq p, \qquad \lim_{n\to\infty} p_n = p. \tag{5}$$

Similarly,

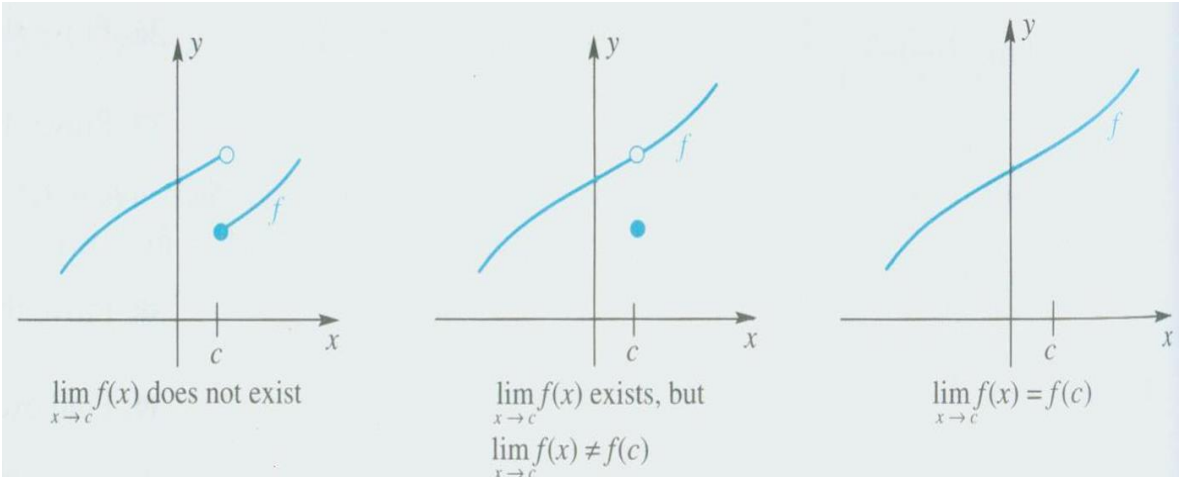
$\lim_{x\to p} f(x) = r$ implies that

$$\lim_{n\to\infty} f(p_n) = r \tag{6}$$

for every sequence $\{p_n\}$ in E such that

$$p_n \neq p, \qquad \lim_{n\to\infty} p_n = p. \tag{7}$$

(5) and (6) says that the sequence $\{f(p_n)\}$ converges to q and r, and by the uniqueness of limit of sequence, we have $q = r$.



(1.4) Definition Let f, g be complex functions defined on E. Then $f+g$, fg and $\frac{f}{g}$ are defined pointwise as follows:

$$(f+g)(x) = f(x)+g(x)$$

$$(fg)(x) = f(x)g(x)$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad \text{provided } g(x) \neq 0.$$

Similarly, if $\mathbf{f}, \mathbf{g}$ map E into $\mathbb{R}^k$, then $\mathbf{f}+\mathbf{g}$, $\mathbf{fg}$ and $\frac{\mathbf{f}}{\mathbf{g}}$ are defined pointwise as follows:

$$(\mathbf{f}+\mathbf{g})(x) = \mathbf{f}(x)+\mathbf{g}(x)$$

$$(\mathbf{f}\cdot\mathbf{g})(x)=\mathbf{f}(x)\cdot\mathbf{g}(x)$$

$$(\lambda\mathbf{f})(x)=\lambda\mathbf{f}(x),\quad \text{where } \lambda \text{ is a real number.}$$

(1.5) Theorem Suppose $E\subset X$, a metric space, p is a limit point of E, f and g are complex functions on E, and

$$\lim_{x\to p} f(x)=A, \qquad \lim_{x\to p} g(x)=B.$$

Then

(*a*) $\lim_{x\to p}(f+g)(x)=A+B;$

(*b*) $\lim_{x\to p}(fg)(x)=AB;$

(*c*) $\lim_{x\to p}\left(\frac{f}{g}\right)(x)=\frac{A}{B},$ if $B\neq 0.$

Proof

(*a*) By Theorem (1.2), $\lim_{x\to p} f(x)=A$ and $\lim_{x\to p} g(x)=B$ implies that

$$\lim_{n\to\infty} f(p_n)=A$$

for every sequence $\{p_n\}$ in E such that

$$p_n\neq p, \qquad \lim_{n\to\infty} p_n=p.$$

and

$$\lim_{n\to\infty} g(p_n)=B$$

for every sequence $\{p_n\}$ in E such that

$$p_n\neq p, \qquad \lim_{n\to\infty} p_n=p.$$

Now

$$\begin{aligned} A+B &= \lim_{n\to\infty} f(p_n)+\lim_{n\to\infty} g(p_n)\\ &= \lim_{n\to\infty}\left[f(p_n)+g(p_n)\right], \text{ by the property of limit of sequences}\\ &= \lim_{n\to\infty}(f+g)(p_n), \text{ by the definition of } f+g \end{aligned}$$

showing again by Theorem (1.2) that $\lim_{x\to p}(f+g)(x)=A+B.$

(*b*) and **(*c*)** can be similarly proved, using the properties of limit of sequences.

(1.6) Remark If **f, g** map E into $\mathbb{R}^k$, then (*a*) remains true, and (*b*) becomes

(b') $\lim_{x\to p}(\mathbf{f}\cdot\mathbf{g})(x) = \mathbf{A}\cdot\mathbf{B}.$

2 Continuous Functions

(2.1) Definition Suppose X and Y are metric spaces, $E \subset X$, $p \in E$, f maps E into Y. Then f is said to be **continuous at the point** p if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$d_Y(f(x), f(p)) < \varepsilon$$

for all points $x \in E$ for which

$$d_X(x, p) < \delta.$$

(2.2) Remark f is said to be **continuous at the point** p if

(i) $f(p)$ is defined i.e., f is defined at the point p;

(ii) $\lim_{x\to p} f(x)$ exists; and

(iii) $\lim_{x\to p} f(x) = f(p)$.

(2.3) Definition If f is continuous at every point of E, then f is said be **continuous on** E.

If p is an isolated point of E, then every function f which has E as its domain of definition is continuous at p. For, no matter which $\varepsilon > 0$ we choose, we can pick $\delta > 0$ so that the only point $x \in E$ for which $d_X(x, p) < \delta.$ is $x = p$; then

$$d_Y(f(x), f(p)) = d_Y(f(p), f(p)) = 0 < \varepsilon.$$

(2.4) Theorem In the situation given in Definition (2.1), assume also that p is a limit point of E. Then f is continuous at p if and only if $\lim_{x\to p} f(x) = f(p)$.

Proof Compare Definitions (1.1) and (2.1).

(2.5) Theorem Suppose X, Y, Z are metric spaces, $E \subset X$, f maps E into Y, g maps the range of f, $f(E)$, into Z, and h is the mapping of E into Z defined by

$$h(x) = g(f(x)) \qquad (x \in E).$$

If f is continuous at a point $p \in E$ and if g is continuous at the point $f(p)$, then h is continuous at p.

i.e., continuous function of a continuous function is continuous.

The function h is called the **composition** or the **composite** of f and g. The notation $h = g \circ f$ is frequently used in this context.

Proof

Let $\varepsilon > 0$ be given. Since g is continuous at $f(p)$, there exists $\eta > 0$ such that

$$d_Z\left(g(y), g\left(f(p)\right)\right) < \varepsilon \tag{8}$$

if $d_Y\left(y, f(p)\right) < \eta$ and $y \in f(E)$.

Since f is continuous at p, corresponding to $\eta > 0$ there exists a $\delta > 0$ such that

$$d_Y(f(x), f(p)) < \eta$$

if $d_X(x, p) < \delta$ and $x \in E$.

Since $h(x) = g(f(x))$ $\left(x \in E\right)$, from (8), we have

$$d_Z\left(h(x), h(p)\right) = d_Z\left(g(f(x)), g\left(f(p)\right)\right) < \varepsilon \tag{9}$$

if $d_X(x, p) < \delta$ and $x \in E$.

Thus h is continuous at p.

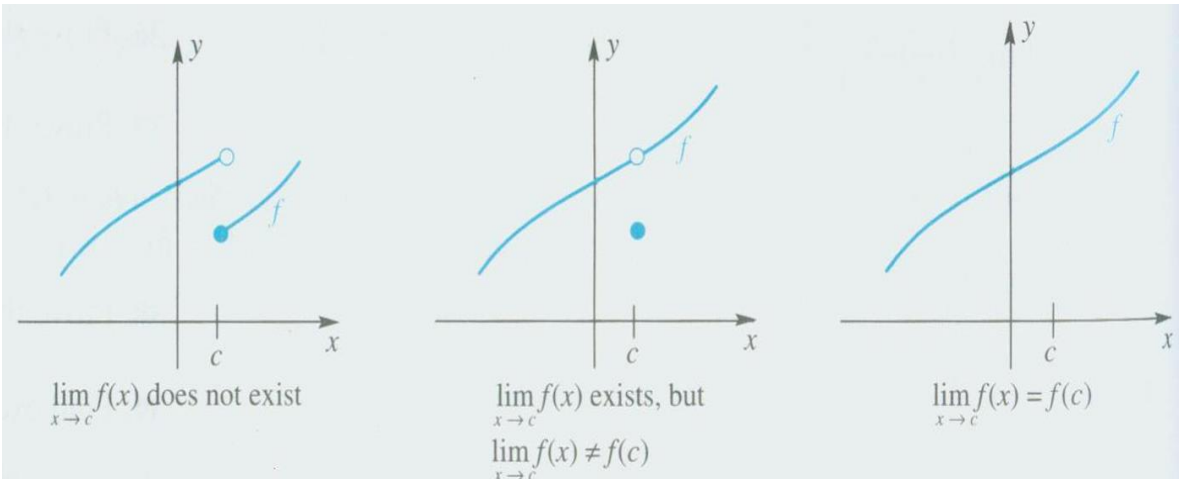


Figure Fig (*a*) and (*b*) show two functions that are discontinuous at $x = c$ but continuous at all other x points. Fig (*c*) gives a continuous function.

(2.6) Theorem (*Characterization of continuity*) A mapping f of a metric space X into a metric space Y is continuous on X if and only if $f^{-1}(V)$ is open in X for every open set V in Y.

Proof

Note that $f^{-1}(V) = \{x \in X \mid f(x) \in V\}$.

Suppose f is continuous on X and V is an open set in Y. To show that $f^{-1}(V)$ is open, it is enough to show that every point of $f^{-1}(V)$ is an interior point of $f^{-1}(V)$. For this, suppose $p \in X$ and $f(p) \in V$. Since V is open, there exists $\varepsilon > 0$ such that $y \in V$ if

$$d_Y(f(p), y) < \varepsilon; \tag{10}$$

and since f is continuous at p, there exists a $\delta > 0$ such that

$$d_Y(f(x), f(p)) < \varepsilon \tag{11}$$

for all points $x \in E$ for which $d_X(x, p) < \delta$.

From (10) and (11), we have $f(x) \in V$ as soon as $d_X(x, p) < \delta$. i.e., $x \in f^{-1}(V)$ as soon as $d_X(x, p) < \delta$. This shows p has a neighborhood $N = \{x \in X \mid d_X(x, p) < \delta\} \subset f^{-1}(V)$; hence p is an interior point of $f^{-1}(V)$. Since p is an arbitrary point of $f^{-1}(V)$, this shows that each point of $f^{-1}(V)$ is an interior point and hence $f^{-1}(V)$ is an open set.

Conversely, suppose $f^{-1}(V)$ is open in X for every open set V in Y. We have to show that f is continuous on X. i.e., to show that f is continuous at every point of X. Fix $p \in X$ and $\varepsilon > 0$, let

$$V = \{y \in Y \mid d_Y(y, f(p)) < \varepsilon\}.$$

Then V, being a neighborhood, is open; hence by assumption $f^{-1}(V)$ is open in X; hence each point of $f^{-1}(V)$ is an interior point, so in particular p is an interior point, so that there exists $\delta > 0$ such that $x \in f^{-1}(V)$ as soon as $d_X(x, p) < \delta$. But if $x \in f^{-1}(V)$, then $f(x) \in V$, so that $d_Y(f(x), f(p)) < \varepsilon$. This shows that f is continuous at p.

This completes the proof.

(2.7) Corollary A mapping f of a metric space X into a metric space Y is continuous if and only if $f^{-1}(C)$ is closed in X for every closed set C in Y.

Proof Corollary follows from the theorem, since a set is closed if and only if its complement is open, and since $f^{-1}(E^C) = [f^{-1}(E)]^C$ for every $E \subset Y$.

(2.8) Theorem Let f and g be complex continuous functions on a metric space X. Then $f+g$, fg, and $\frac{f}{g}$ are continuous on X. (In the last case, we must of course assume that $g(x) \neq 0,\ \forall x \in X.$)

Proof

Case 1) At isolated points of X there is nothing to prove.

Case 2) At limit points, the statement follows from Theorems (1.5)and (2.4). We prove only one and leaving the other to exercises.

By Theorem (2.4), f and g continuous functions implies $\lim_{x\to p} f(x) = f(p)$ and $\lim_{x\to p} g(x) = g(p)$. To show that $f+g$ is continuous, again by Theorem (2.4), it is enough to show that $\lim_{x\to p}(f+g)(x) = (f+g)(p)$.

Now
$$\lim_{x\to p}(f+g)(x) = \lim_{x\to p}[f(x)+g(x)], \text{ by the definition of } f+g$$
$$= \lim_{x\to p} f(x) + \lim_{x\to p} g(x), \text{ by Theorem (1.5)}$$
$$= f(p)+g(p)$$
$$= (f+g)(p), \text{ by the definition of } f+g$$

(2.9) Theorem

(*a*) Let $f_1, \ldots, f_k$ be real functions on a metric space X, and let $\mathbf{f}$ be the mapping of X into $\mathbb{R}^k$ defined by

$$\mathbf{f}(x) = (f_1(x), \ldots, f_k(x)) \quad (x \in X);$$

then $\mathbf{f}$ is continuous if and only if each of the functions $f_1, \ldots, f_k$ is continuous.

(*b*) If $\mathbf{f}$ and $\mathbf{g}$ are continuous mappings of X into $\mathbb{R}^k$, then $\mathbf{f}+\mathbf{g}$ and $\mathbf{f}\cdot\mathbf{g}$ are continuous on X.

The functions $f_1, \ldots, f_k$ are called the **components** of $\mathbf{f}$. Note that $\mathbf{f}+\mathbf{g}$ is a mapping into $\mathbb{R}^k$, whereas $\mathbf{f}\cdot\mathbf{g}$ is **real function** on X.

Proof Part (*a*) follows from the inequalities

$$|f_j(x) - f_j(y)| \leq |\mathbf{f}(x) - \mathbf{f}(y)| = \left\{\sum_{i=1}^{k} |f_i(x) - f_i(y)|^2\right\}^{1/2},$$

for $j = 1, \ldots, k$. Part (*b*) follows from (*a*) and Theorem (2.8).

(2.10) Examples If $x_1, \ldots, x_k$ are the coordinates of the point $\mathbf{x} \in \mathbb{R}^k$, the functions ϕ_i defined by

$$\phi_i(\mathbf{x}) = x_i \quad \left(\mathbf{x} \in \mathbb{R}^k\right)$$

are continuous on $\mathbb{R}^k$, since the inequality

$$\left|\phi_i(\mathbf{x}) - \phi_i(\mathbf{y})\right| \leq \left|\mathbf{x} - \mathbf{y}\right|$$

shows that we may take $\delta = \varepsilon$ in Definition (2.1). The functions ϕ_i are some times called the **coordinate functions.**

Repeated application of Theorem (2.8) then shows that every monomial

$$x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} \tag{12}$$

where $n_1, \ldots, n_k$ are nonnegative integers, is continuous on $\mathbb{R}^k$. The same is true of constant multiples of (12), since constants are evidently continuous. It follows that every polynomial P, given by

$$P(\mathbf{x}) = \sum c_{n_1, \ldots, n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} \qquad \left(\mathbf{x} \in \mathbb{R}^k\right), \tag{13}$$

is continuous on $\mathbb{R}^k$. Here the coefficients $c_{n_1, \ldots, n_k}$ are complex numbers, $n_1, \ldots, n_k$ are nonnegative integers, and the sum in (13) has finitely many terms.

Furthermore, every rational function in $x_1, \ldots, x_k$, that is, every quotient of two polynomials of the form (13), is continuous on $\mathbb{R}^k$ wherever the denominator is different from zero.

From the triangle inequality one sees easily that

$$\left|\left|\mathbf{x}\right| - \left|\mathbf{y}\right|\right| \leq \left|\mathbf{x} - \mathbf{y}\right| \qquad \left(\mathbf{x}, \mathbf{y} \in \mathbb{R}^k\right). \tag{14}$$

Hence the mapping $\mathbf{x} \to \left|\mathbf{x}\right|$ is a continuous real function on $\mathbb{R}^k$.

If now $\mathbf{f}$ is a continuous mapping from a metric space X into $\mathbb{R}^k$, and if ϕ is defined on X by setting $\phi(p) = \left|\mathbf{f}(p)\right|$, it follows, by Theorem (2.5), that ϕ is a continuous real function on X.

3 Continuity and Compactness

(3.1) Definition A mapping $\mathbf{f}$ of a set E into $\mathbb{R}^k$ is said to be **bounded** if there is a real number M such that $\left|\mathbf{f}(x)\right| \leq M \quad \forall x \in E$.

(3.2) Theorem Suppose f is a continuous mapping of a compact metric space X into a metric space Y. Then $f(X)$ is compact.

Proof

Let $\{V_\alpha\}$ be an open cover of $f(X)$. Since f is continuous, Theorem (2.6) shows that each of the sets $f^{-1}(V_\alpha)$ is open. Since X is compact, the open cover $\left\{f^{-1}(V_\alpha)\right\}$ has a finite subcover consisting of $f^{-1}(V_{\alpha_1}), \ldots, f^{-1}(V_{\alpha_n})$, such that

$$X \subset f^{-1}(V_{\alpha_1}) \cup \ldots \cup f^{-1}(V_{\alpha_n}). \tag{15}$$

Since $f\left(f^{-1}(E)\right) \subset E$ for every $E \subset Y$, (15) implies that

$$f(X) \subset V_{\alpha_1} \cup \ldots \cup V_{\alpha_n}, \tag{16}$$

showing that $f(X)$ can be covered by a finite subcover of the given cover $\{V_\alpha\}$. Hence $f(X)$ is compact.

(3.3) Theorem If **f** is a continuous mapping of a compact metric space X into $\mathbb{R}^k$, then $\mathbf{f}(X)$ is closed and bounded. Thus, **f** is bounded.

Proof By the previous Theorem, $\mathbf{f}(X)$ is compact. We know that a compact set in $\mathbb{R}^k$ is closed and bounded. Hence $\mathbf{f}(X)$ is closed and bounded. In particular, **f** is bounded.

(3.4) Theorem Suppose f is a continuous real function on a compact metric space X, and

$$M = \sup\{f(p) \mid p \in X\}, \qquad m = \inf\{f(p) \mid p \in X\}. \tag{16}$$

Then there exists points $a, b \in X$ such that $f(a) = M$ and $f(b) = m$.

(Note that in the above M is the least upper bound of the set $\{f(p) \mid p \in X\}$ and m is the greatest lower bound of the same set.)

The conclusion of the theorem may also be stated as follows:

" There exist points a and b in X such that $f(a) \le f(x) \le f(b)\ \forall x \in X$; that is, f attains its maximum at a and minimum at b. "

Proof

By Theorem (3.3), $f(X)$ is a closed and bounded set of real numbers; hence by supremum and infimum properties of $\mathbb{R}$, supremum and infimum exists for the bounded set $f(X) = \{f(p) \mid p \in X\}$ and since $f(X)$ is closed the supremum M and infimum m are

elements of $f(X)=\{f(p)\mid p\in X\}$. Hence there exist elements $a, b\in X$ such that $f(a)=M$ and $f(b)=m$.

(3.5) Theorem Suppose f is a continuous 1-1 mapping of a compact metric space X into a metric space *Y*. Then the inverse mapping f^{-1} defined on Y by

$$f^{-1}(f(x))=x \qquad (x\in X)$$

is a continuous mapping of Y onto X.

Proof

Applying Theorem (2.6) to f^{-1} in place of f, we see that it suffices to prove that $f(V)$ is an open set in Y for every open set V in X. Fix such a set V.

The complement V^C of the open set V is closed in X, hence compact (As "closed subsets of compact sets are compact"); hence, by Theorem (3.2), $f(V^C)$ is a compact subset of Y and so is closed in *Y* (As "compact subsets of metric spaces are closed"). Since f is one-to-one and onto, $f(V)$ is the complement of $f(V^C)$. Hence $f(V)$ is open, proving that f^{-1} is a continuous mapping of Y onto X.

(3.6) Definition Let f be a mapping of a metric space X into a metric space Y. We say that f is **uniformly continuous on** X if for every $\varepsilon>0$ there exists a $\delta>0$ such that

$$d_Y(f(p), f(q))<\varepsilon \tag{16}$$

for all points $p, q\in X$ for which $d_X(p,q)<\delta$.

Differences between *continuity* and *uniform continuity*

- Uniform continuity is a property of a function on set, whereas continuity can be defined at single point. To ask whether a given function is uniformly continuous at a certain point is meaningless.
- If f is continuous on X, then it is continuous at each point of X, so it is possible to find, for each $\varepsilon>0$ and for each point $p\in X$, a number $\delta>0$ having the property specified in Definition (2.1). **This δ depends on ε and on p.** If f is however, unifiormly continuous on X, then it is possible, for each $\varepsilon>0$, to find **one** number $\delta>0$ which will do for **all** points p of X. **This δ depends on ε only.**

(3.7) Theorem Let f be a continuous mapping of a compact metric space *X* into a metric space *Y*. Then f is uniformly continuous on X. (UQ 2006)

Proof Let $\varepsilon > 0$ be given. Since f is continuous, we can associate to each point $p \in X$ a positive number $\phi(p)$ such that

$$q \in X,\ d_X(p,q) < \phi(p) \quad \text{implies} \quad d_Y(f(p), f(q)) < \frac{\varepsilon}{2}. \tag{16}$$

Let $J(p)$ be the set of all $q \in X$ for which

$$d_X(p,q) < \frac{1}{2}\phi(p). \tag{17}$$

Since $p \in J(p)$, the collection of all sets $J(p)$ is an open cover of X; and since X is compact, there is a finite set of points $p_1, \ldots, p_n$ in X, such that

$$X \subset J(p_1) \cup \cdots \cup J(p_n). \tag{18}$$

We put

$$\delta = \frac{1}{2}\min\left[\phi(p_1), \cdots, \phi(p_n)\right]. \tag{19}$$

Then $\delta > 0$. (This is one point where the finiteness of the covering, inherent in the definition of compactness, is essential. The minimum of a finite set of positive numbers is positive, whereas the infimum of an infinite set of positive numbers may very well be 0; for example $\inf\left\{\frac{1}{n} : n \in \square\right\} = 0.$)

Now let q and p be points of X, such that $d_X(p,q) < \delta$. By (18), there is an integer m, $1 \le m \le n$, such that $p \in J(p_m)$; hence

$$d_X(p, p_m) < \frac{1}{2}\phi(p_m), \tag{20}$$

and we also have

$$d_X(q, p_m) < d_X(p,q) + d_X(p, p_m) < \delta + \frac{1}{2}\phi(p_m) \le \phi(p_m).$$

Finally, (16) shows that therefore

$$d_Y(f(p), f(q)) \le d_Y(f(p), f(p_m)) + d_Y(f(q), f(p_m)) < \varepsilon.$$

This completes the proof.

(3.8) Theorem Let E be a noncompact set in $\square^1$. Then

(a) there exists a continuous function on E which is not bounded;

(b) there exists a continuous and bounded function on E which has no maximum.

If, in addition, E is bounded, then

(c) there exists a continuous function on E which is not uniformly continuous.

Proof

Suppose first that E is bounded.

(*a*) Since E is not compact, E bounded implies that E is not closed, so that there exists a limit point x_0 of E which is not a point of E. Consider

$$f(x) = \frac{1}{x - x_0} \qquad (x \in E). \tag{21}$$

This is continuous on E, since the constant function 1 and $x - x_0$ are continuous and their quotient is also continuous by Theorem (2.8). But f is unbounded.

(*b*) To see that (21) is not uniformly continuous, let $\varepsilon > 0$ and $\delta > 0$ be arbitrary, and choose a point $x \in E$ such that $|x - x_0| < \delta$. Taking t close enough to x_0, we can then make the difference $|f(t) - f(x)|$ greater than ε, although $|t - x| < \delta$. Since this is true for every $\delta > 0$, f is uniformly continuous on E.

(*c*) The function g defined by

$$g(x) = \frac{1}{1 + (x - x_0)^2} \qquad (x \in E) \tag{22}$$

is continuous on E, and is bounded, since $0 < g(x) < 1$. It is clear that

$$\sup\{g(x) \mid x \in E\} = 1,$$

whereas $g(x) < 1$ for all $x \in E$. Thus g has no maximum on E.

Now suppose that E is unbounded.

(*a*) Then $f(x) = x$ establishes (*a*).

(*b*) The function h defined by

$$h(x) = \frac{x^2}{1 + x^2} \qquad (x \in E) \tag{23}$$

establishes (*b*), since

$$\sup\{h(x) \mid x \in E\} = 1$$

and $h(x) < 1$ for all $x \in E$.

Assertion (*c*) would be false if boundedness were omitted from the hypotheses. For, let E be the set of all integers. Then every function defined on E is uniformly continuous on E. To see this, we need merely take $\delta < 1$ in Definition (3.6).

(3.9) Example Let X be the half-open interval $[0, 2\pi)$ on the real line, and let $\mathbf{f}$ be the mapping of X onto the circle Y consisting of all points whose distance from the origin is 1, given by

$$\mathbf{f}(t) = (\cos t, \sin t) \qquad (0 \le t < 2\pi). \tag{24}$$

The trigonometric functions cosine and sine are continuous. Also, they are periodic with period 2π. These results show that **f** is a continuous 1-1 mapping of X onto Y.

However, the inverse mapping (which exists, since **f** is one-to-one and onto) fails to be continuous at the point (1, 0) = **f**(0). Of course, X is not compact in this example.

Another argument to the fact that $\mathbf{f}^{-1}$ is not continuous, for if : $\mathbf{f}^{-1}$ is continuous, then $\mathbf{f}^{-1}(Y)$ is compact implies $X = [0, 2\pi)$ is compact which is false.

4Continuity and Connectedness

(4.1) Theorem If f is a continuous mapping of a metric space X into a metric pace Y, and if E is a connected subset of X, then $f(E)$ is connected.

Proof

Assume, on the contrary, that $f(E)$ is not connected. Then $f(E) = A \cup B$, where A and B are nonempty separated subsets of Y. Put $G = E \cap f^{-1}(A)$, $H = E \cap f^{-1}(B)$.

Then $E = G \cup H$, and neither G nor H is empty.

Since $A \subset \bar{A}$ (the closure of A), we have $G \subset f^{-1}(\bar{A})$; the latter set is closed, since f is continuous; hence $\bar{G} \subset f^{-1}(\bar{A})$. It follows that $f(\bar{G}) \subset \bar{A}$. Since $f(H) = B$ and $\bar{A} \cap B = \Phi$, we conclude that $\bar{G} \cap H = \Phi$.

The same argument shows that $G \cap \bar{H} = \Phi$. Thus G and H are separated and implies that E is not connected, contrary to the fact that E is connected.

We need the following theorem A for proving Theorem (4.2).

Theorem A: A subset E of the real line $\mathbb{R}^1$ is connected if and only if it has the following property: If $x \in E$, $y \in E$, and $x < z < y$, then $z \in E$.

(4.2) Theorem (*Intermediate value theorem*) Let f be a continuous real function on the interval $[a, b]$. If $f(a) < f(b)$ and if c is a number such that $f(a) < c < f(b)$, then there exists a point $x \in (a, b)$ such that $f(x) = c$.

A similar result holds, of course, if $f(a) > f(b)$. Roughly speaking, the theorem says that a continuous real function assumes all intermediate values on an interval.

Proof By Theorem *A* above, $[a,b]$ is connected; hence by Theorem (4.1) shows that $f([a,b])$ is connected subset of $\square^1$, and the assertion follows if we appeal once more to Theorem *A*.

(4.3) Theorem At first glance, it might seem that Theorem (4.2) has a converse. That is, one might think that if for any two points $x_1 < x_2$ and for any number c between $f(x_1)$ and $f(x_2)$ there is a point $x \in (x_1, x_2)$ such that $f(x) = c$, then f must be continuous (Refer the example in the coming section.

5 Discontinuity

(5.1) Definition Let f be defined on (a,b). Consider any point x such that $a \le x < b$. We define $f(x+)$, **the right hand limit at** *x*, by

$$f(x+) = q$$

if $f(t_n) \to q$ as $n \to \infty$, for all sequences $\{t_n\}$ in (x,b) such that $t_n \to x$.

(5.2) Definition Let f be defined on (a,b). Consider any point x such that $a < x \le b$. We define $f(x-)$, **the left hand limit at** *x*, by

$$f(x-) = r$$

if $f(t_n) \to r$ as $n \to \infty$, for all sequences $\{t_n\}$ in (a,x) such that $t_n \to x$.

(5.3) Remark Let f be defined on (a,b). Consider any point x such that $a \le x \le b$. It can be seen that $\lim_{t \to x} f(t)$, **the limit of** f **at** *x*, exists if and only if

$$f(x+) = f(x-) = \lim_{t \to x} f(t).$$

(5.4) Definition Let f be defined on (a,b). If f is discontinuous at a point x, and if $f(x+)$ and $f(x-)$ exist, then f is said to have a **discontinuity of the first kind,** or a **simple discontinuity,** at *x*. Otherwise the discontinuity is said to be of the **second kind.**

There are two ways in which a function can have a simple discontinuity: either $f(x+) \ne f(x-)$ (in which case the value of $f(x)$ is not considered), or $f(x+) = f(x-) \ne f(x)$.

(5.5) Examples

***Example* 1** Define f by

$$f(x)=\begin{cases}1 & \text{when } x \text{ rational}\\ 0 & \text{when } x \text{ irrational}\end{cases}$$

Then at any point x, neither $f(x+)$ nor $f(x-)$ exist, hence f has a **discontinuity of the second kind at every point *x*.**

***Example* 2** Define f by

$$f(x)=\begin{cases}x+2 & \text{when } -3<x<-2\\ -x-2 & \text{when } -2\le x<0\\ x+2 & \text{when } 0\le x<1\end{cases}$$

At the point $x = 0$, $f(0+)=2$ while $f(0-)=-2$ exist, hence $f(0+)$ and $f(0-)$ exist, but they are not equal. Hence f has a **simple discontinuity at** $x = 0$. **But** f is **continuous** at every *other* point of $(-3,1)$.

***Example* 3** Define f by

$$f(x)=\begin{cases}\sin\dfrac{1}{x} & \text{when } x\neq 0\\ 0 & \text{when } x=0\end{cases}$$

At the point $x = 0$, neither $f(0+)$ nor $f(0-)$ exist, hence f has a **discontinuity of the second kind at** $x = 0$. **But** f is **continuous** at every *other* than $x = 0$, since $\sin y$ is a continuous function and $\dfrac{1}{x}$ when $x\neq 0$ is also continuous function, hence their composition $\sin\dfrac{1}{x}$ is continuous at $x\neq 0$.

6 Monotonic Functions

(6.1) Definition Let f be real on (a,b). Then f is said to be **monotonically increasing** on (a,b) if $a<x<y<b$ implies $f(x)\le f(y)$.

(6.2) Definition Let f be real on (a,b). Then f is said to be **monotonically decreasing** on (a,b) if $a<x<y<b$ implies $f(x)\ge f(y)$.

(6.3) Theorem Let f be monotonically increasing on (a,b). Then $f(x+)$ and $f(x-)$ exist at every point $x\in(a,b)$. More precisely,

$$\sup\{f(t): t \in (a,x)\} = f(x-) \le f(x) \le f(x+) = \inf\{f(t): t \in (x,b)\}. \qquad (1)$$

Furthermore, if $a < x < y < b$, then

$$f(x+) \le f(y-). \qquad (2)$$

(UQ 2006)

Proof

Since f is monotonically increasing, the set $\{f(t): a < t < x\}$ is bounded above by $f(x)$. Therefore, by the supremum property of the real line, $\{f(t): a < t < x\}$ has a least upper bound which we shall denote by A. i.e., $A = \sup\{f(t): a < t < x\}$. Clearly,

$$A \le f(x).$$

<u>*Claim 1:*</u> $A = f(x-)$. (i.e., the left hand limit of f at x is A).

Let $\varepsilon > 0$ be given. It follows from the definition of A as a least upper bound that there exists $\delta > 0$ such that $a < x - \delta < x$ and

$$A - \varepsilon < f(x-\delta) \le A. \qquad (3)$$

Since f is monotonic increasing, we have

$$f(x-\delta) \le f(t) \le A \qquad (x-\delta < t < x). \qquad (4)$$

Combining (3) and (4), we see that

$$|f(t) - A| < \varepsilon \qquad (x-\delta < t < x).$$

Hence $f(x-) = A$, and the claim is proved.

Since f is monotonically increasing, the set $\{f(t): x < t < b\}$ is bounded below by $f(x)$. Therefore, by the infimum property of the real line, $\{f(t): x < t < b\}$ has a greatest lower bound which we shall denote by B. i.e., $B = \inf\{f(t): x < t < b\}$. Clearly,

$$f(x) \le B.$$

<u>*Claim 2:*</u> $B = f(x+)$. (i.e., the right hand limit of f at x is B).

Let $\varepsilon > 0$ be given. It follows from the definition of B as a greatest lower bound that there exists $\delta > 0$ such that $x < x + \delta < b$ and

$$B \le f(x+\delta) < B + \varepsilon. \qquad (5)$$

Since f is monotonic increasing, we have

$$B \le f(t) \le f(x+\delta). \qquad (x < t < x+\delta). \qquad (6)$$

Combining (5) and (6), we see that

$$|f(t)-B|<\varepsilon \qquad (x<t<x+\delta).$$

Hence $f(x+)=B,$ and the claim is proved.

Next, if $a<x<y<b$, we see from (1) that

$$f(x+)=\inf\{f(t):t\in(x,b)\}=\inf\{f(t):t\in(x,y)\}. \tag{7}$$

In the above, the last equality is obtained by applying (1) to (a,y) in place of (a,b). Similarly,

$$f(y-)=\sup\{f(t):t\in(a,y)\}=\sup\{f(t):t\in(x,y)\}. \tag{8}$$

Comparison of (7) and (8) gives (2).

(6.4) Theorem Let f be monotonically decreasing on (a,b). Then $f(x+)$ and $f(x-)$ exist at every point $x\in(a,b)$. More precisely,

$$\inf\{f(t):t\in(a,x)\}=f(x-)\geq f(x)\geq f(x+)=\sup\{f(t):t\in(x,b)\}.$$

Furthermore, if $a<x<y<b$, then

$$f(x+)\geq f(y-).$$

Proof is left to the exercise.

(6.5) Corollary Monotonic functions have no discontinuities of the second kind.

(6.6) Remark The above Corollary implies that every monotonic function is discontinuous at a countable number of points at most. A simpler proof is given in the following Theorem.

(6.7) Theorem Let f be monotonic on (a,b). Then the set of points of (a,b) at which f is discontinuous is at most countable. (UQ 2006)

Proof Suppose, for the sake of definiteness, that f is increasing, and let

$$E=\{x\in(a,b):f \text{ is discontinuous at } x\}.$$

Claim: E is countable.

Note that x is point of E implies that x is a point at which f is discontinuous. By Corollary (6.5) f has no discontinuities of the second kind. Hence the discontinuity at x is the discontinuity of the first kind. Hence two cases arise:

Case 1) $f(x+)\neq f(x-)$

or Case 2) $f(x+)=f(x-)\neq f(x).$

The second case is not possible, since f is monotonic increasing (Think yourself!). Hence at the discontinuous point x, we have $f(x+) \neq f(x-)$.

Hence with every point we associate a rational number $r(x)$ such that

$$f(x-) < r(x) < f(x+).$$

Since $x_1 < x_2$ implies $f(x_1+) \leq f(x_2-)$, we see that $r(x_1) \neq r(x_2)$ if $x_1 \neq x_2$.

We have thus established a 1-1 correspondence between the set E and a subset of the set of rational numbers. Since the set of rational numbers is countable, this shows that E is countable. i.e., the set of points of (a,b) at which f is discontinuous is at most countable.

7 Infinite Limits and Limits at Infinity

(7.1) Definition For any real c, the set of real numbers x such that $x > c$ is called a **neighborhood of** $+\infty$ and is written $(c, +\infty)$. Similarly, $(-\infty, c)$ is a neighborhood of $-\infty$.

(7.2) Definition Let f be a real function defined on E. We say that

$$f(t) \to A \text{ as } t \to x,$$

where A and x are extended real number system, if for every neighbourhood U of A there is a neighborhood V of x such that $V \cap E \neq \Phi$, and such that $f(t) \in U \ \forall t \in V \cap E,\ t \neq x$.

(7.3) Theorem Let f and g be defined on E. Suppose

$$f(t) \to A, \quad g(t) \to B \quad \text{as } t \to x.$$

Then

(a) $f(t) \to A'$ implies $A' = A$.

(b) $(f+g)(t) \to A+B$.

(c) $(fg)(t) \to AB$.

(d) $\left(\frac{f}{g}\right)(t) \to \frac{A}{B}$,

provided the right members of (*b*), (*c*), and (*d*) are defined.

Note that $\infty - \infty$, $0 \cdot \infty$, $\frac{\infty}{\infty}$, $\frac{A}{0}$ are not defined.

Chapter 4

DIFFERENTIATION

1 The Derivative of a Real Function

(1.1) Definition Let f be defined (and real-valued) on $[a,b]$. For any $x \in [a,b]$ form the quotient

$$\phi(t) = \frac{f(t) - f(x)}{t - x} \qquad (a < t < b,\ t \neq x), \tag{1}$$

and define

$$f'(x) = \lim_{t \to x} \phi(t), \tag{2}$$

provided this limit exists in accordance with Definition 1.1 in Chapter 3 *Continuity.*

We thus associate with the function f a function f' whose domain is the set of points x at which the limit (2) exists; f' is called the **derivative of f**. In particular $f'(x)$ is the derivative of f at x.

If f' is defined at a point x, we say that f is **differentiable at x.** If f' is defined at every point of a set $E \subset [a, b]$, we say that f is differentiable on E.

(1.2) Definition/Remark

- In (2), if we consider $\phi(x+)$, the right hand limit of ϕ at x, in place of $\lim_{t \to x} \phi(t)$, then

  $$f'(x+) = \lim_{t \to x+} \phi(t), \tag{3}$$

  and $f'(x+)$ is the **right hand derivative** of f at the point x.

- In (2), if we consider $\phi(x-)$, the left hand limit of ϕ at x, in place of $\lim_{t \to x} \phi(t)$, then

  $$f'(x-) = \lim_{t \to x-} \phi(t), \tag{4}$$

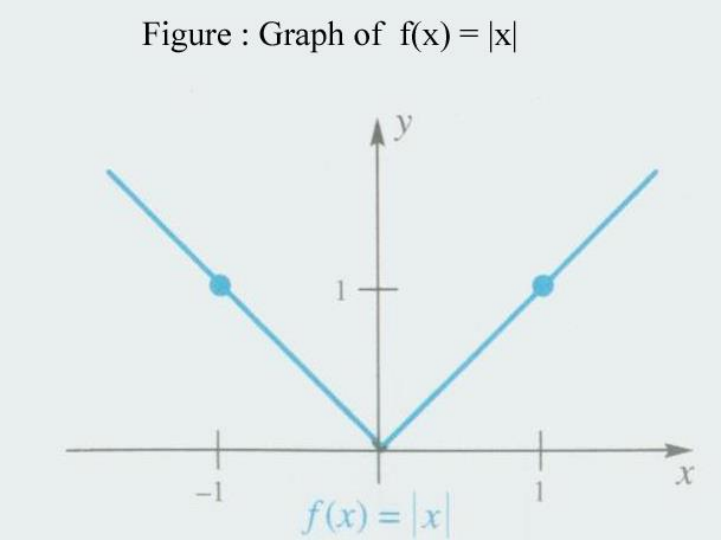
  and $f'(x-)$ is the **left hand derivative** of f at the point x.

- At the end points a and b, the derivative, if it exists, is a right-hand or left-hand derivative, respectively.

- If f is defined on a segment (a, b) and if $a < x < b$, then $f'(x)$ is defined by (1) and (2), as above. But $f'(a)$ and $f'(b)$ are not defined in this case.

(1.3) Example Consider the function $f(x)=|x|$ defined over $\square$. The left hand limit at $x = 0$ is –1, while the right hand limit is 1. As they are different, f is not differentiable at $x = 0$. For the points $x<0$, both left and right hand limits coincide and the value is -1. Hence the derivative of f at $x<0$ is –1. Similarly, it can be seen that f is differentiable for the points $x>0$, and the derivative at each of those points is 1. i.e., f differentiable at all points except $x = 0$.

Figure : Graph of f(x) = |x|



(1.4) Theorem Let f be defined on $[a,b]$. If f is differentiable at a point $x\in[a,b]$, then f is continuous at *x*.

Proof

Note that $$f(t)-f(x)=\frac{f(t)-f(x)}{t-x}(t-x).$$

$$\lim_{t\to x} f(t)-f(x)=\lim_{t\to x}\left[\frac{f(t)-f(x)}{t-x}(t-x)\right]$$

$$=\lim_{t\to x}\frac{f(t)-f(x)}{t-x}\lim_{t\to x}(t-x), \text{ by Theorem (1.5) of Chapter 3}$$

$$=f'(x)\times 0$$

$$=0.$$

Hence $$\lim_{t\to x} f(t)=f(x),$$

showing that f is continuous at *x*.

(1.5) Theorem Suppose f and g defined on $[a,b]$ and are differentiable at a point $x\in[a,b]$. Then $f+g$, fg, and $\frac{f}{g}$ are differentiable at *x*, and

(a) $(f+g)'(x)=f'(x)+g'(x)$;

(b) $(fg)'(x)=f'(x)g(x)+f(x)g'(x)$;

$$(c)\ \left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - g'(x)f(x)}{g^2(x)},\ \text{provided}\ g(x) \neq 0.$$

Proof

(*a*)
$$(f+g)'(x) = \lim_{t\to x}\frac{(f+g)(t) - (f+g)(x)}{t-x}$$
$$= \lim_{t\to x}\frac{f(t)-f(x)}{t-x} + \lim_{t\to x}\frac{g(t)-g(x)}{t-x},$$
by Theorem (1.5) of Chapter 3
$$= f'(x) + g'(x).$$

(*b*) Let $h = fg$. Then

$$h(t) - h(x) = f(t)\big[g(t) - g(x)\big] + g(x)\big[f(t) - f(x)\big].$$

Dividing the above by $t - x$, we obtain

$$\frac{h(t)-h(x)}{t-x} = f(t)\left[\frac{g(t)-g(x)}{t-x}\right] + g(x)\left[\frac{f(t)-f(x)}{t-x}\right].$$

Hence

$$h'(x) = \lim_{t\to x}\frac{h(t)-h(x)}{t-x} = \lim_{t\to x} f(t)\lim_{t\to x}\left[\frac{g(t)-g(x)}{t-x}\right] + \lim_{t\to x} g(x)\lim_{t\to x}\left[\frac{f(t)-f(x)}{t-x}\right]$$
$$= f(x)g'(x) + g(x)f'(x).$$

i.e., $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$ is proved.

(*c*) Let $h = \frac{f}{g}$. Then

$$h(t) - h(x) = \frac{f(t)}{g(t)} - \frac{f(x)}{g(x)} = \frac{f(t)g(x) - f(x)g(t)}{g(t)g(x)}$$
$$= \frac{g(x)\big[f(t)-f(x)\big] - f(x)\big[g(t)-g(x)\big]}{g(t)g(x)}$$

Hence

$$\frac{h(t)-h(x)}{t-x} = \frac{1}{g(t)g(x)}\left\{g(x)\left[\frac{f(t)-f(x)}{t-x}\right] - f(x)\left[\frac{g(t)-g(x)}{t-x}\right]\right\}.$$

Hence

$$h'(x) = \lim_{t\to x} \frac{h(t)-h(x)}{t-x}$$

$$= \lim_{t\to x} \frac{1}{g(t)g(x)} \left\{ \lim_{t\to x} g(x) \lim_{t\to x} \left[\frac{f(t)-f(x)}{t-x} \right] - \lim_{t\to x} f(x) \lim_{t\to x} \left[\frac{g(t)-g(x)}{t-x} \right] \right\}$$

$$= \frac{g(x)f'(x) - g'(x)f(x)}{g^2(x)}$$

(1.6) Examples

***Example* 1** The derivative of any constant function is clearly 0.

***Example* 2** If f is defined by

$$f(x) = x,$$

then

$$f'(x) = 1.$$

***Example* 3** Let n be a non-negative integer. If f is defined by $f(x) = x^n$, then repeated application of (*b*) in Theorem above shows that f is differentiable and $f'(x) = nx^{n-1}$.

***Example* 4** Let n be a negative integer. If f is defined by $f(x) = x^n$, $(x \neq 0)$ then repeated application of (*c*) above shows that f is differentiable and $f'(x) = nx^{n-1}$.

> **Remark** In view Examples 3 and 4 above, **every polynomial is differentiable**, **and so is every rational function,** except at the points where the denominator is zero.

(1.7) Theorem ***(Chain rule for differentiation)*** Suppose f is continuous on $[a,b]$, $f'(x)$ exists at some point $x \in [a,b]$, g is defined on an interval I which contains the range of f, and g is differentiable at the point $f(x)$. If

$$h(t) = g(f(t)) \qquad (a \le t \le b),$$

then h is differentiable at x, and

$$h'(x) = g'(f(x))f'(x). \tag{5}$$

Proof

Let $y = f(x)$. By the definition of the derivative,

$$f'(x) = \lim_{t\to x} \frac{f(t)-f(x)}{t-x}.$$

Hence

$$f(t) - f(x) = (t-x)[f'(x) + u(t)], \tag{6}$$

where $t \in [a,b]$, and $u(t) \to 0$ as $t \to x$

Similarly,

$$g(s) - g(y) = (s - y)[g'(y) + v(s)], \tag{7}$$

where $s \in I$, $v(s) \to 0$ as $s \to y$.

Let $s = f(t)$.

Then

$$\begin{aligned} h(t) - h(x) &= g(f(t)) - g(f(x)) \\ &= [f(t) - f(x)][g'(y) + v(s)], \text{ using (7)} \\ &= (t - x)[f'(x) + u(t)][g'(y) + v(s)], \text{ using (6)} \end{aligned}$$

Hence if $t \neq x$,

$$\frac{h(t) - h(x)}{t - x} = [g'(y) + v(s)][f'(x) + u(t)]. \tag{8}$$

Letting $t \to x$, we see that $s \to y$, by the continuity of f, so that the right side of (8) tends to $g'(y) f'(x)$, which gives (5).

(1.8) Examples

***Example* 1** Let f be defined by

$$f(x) = \begin{cases} x \sin \dfrac{1}{x} & (x \neq 0) \\ 0 & (x = 0) \end{cases}$$

Noting that the derivative of $\sin x$ is $\cos x$. Also note, by the chain rule for differentiation (Theorem (1.7)), that the derivative for $x \neq 0$ of $\sin \dfrac{1}{x}$ is $\cos \dfrac{1}{x} \times \left(-\dfrac{1}{x^2} \right)$. Applying Theorem (1.5), we obtain

$$f'(x) = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x} \qquad (x \neq 0). \tag{9}$$

At $x = 0$, these Theorems do not apply any longer, since $\sin \dfrac{1}{x}$ is not defined there, and we appeal directly to the definition: for $t \neq 0$,

$$\frac{f(t) - f(0)}{t - 0} = \sin \frac{1}{t}.$$

As $t \to 0$, this does not tend to any limit, so that $f'(0)$ does not exist.

***Example* 2** Let f be defined by

$$f(x)=\begin{cases} x^2\sin\dfrac{1}{x} & (x\neq 0) \\ 0 & (x=0) \end{cases}$$

As above, applying Theorems (1.7) and (1.5), we obtain

$$f'(x)=2x\sin\frac{1}{x}-\cos\frac{1}{x} \qquad (x\neq 0). \tag{10}$$

At $x=0$, we appeal directly to the definition: for $t\neq 0$,

$$\left|\frac{f(t)-f(0)}{t-0}\right|=\left|t\sin\frac{1}{t}\right|\leq|t|.$$

As $t\to 0$, we see that

$$f'(0)=0.$$

Thus f is differentiable at all points x, but f' is not a continuous function, since $\cos\dfrac{1}{x}$ in (10) does not tend to a limit as $x\to 0$, so that $\lim\limits_{x\to 0} f'(x)$ doesn't exist.

2 Mean Value Theorems

(2.1) Definition Let f be a real function defined on a metric space X. We say that f has a **local maximum** at a point $p\in X$ if there exists $\delta>0$ such that $f(q)\leq f(p)\ \ \forall q\in X$ with $d(p,q)<\delta$.

(2.2) Definition Let f be a real function defined on a metric space X. We say that f has a **local minimum** at a point $p\in X$ if there exists $\delta>0$ such that $f(q)\geq f(p)\ \ \forall q\in X$ with $d(p,q)<\delta$.

(2.3) Theorem Let f be defined on $[a,b]$; if f has a local maximum at a point $x\in(a,b)$, and if $f'(x)$ exists, then $f'(x)=0$.

The analogous statement for local minima is "Let f be defined on $[a,b]$; if f has a local minimum at a point $x\in(a,b)$, and if $f'(x)$ exists, then $f'(x)=0$" and is true.

Proof of the Theorem

Choose $\delta>0$ in accordance with Definition (2.1), so that

$$a<x-\delta<x<x+\delta<b.$$

If $x-\delta<t<x$, then

$$\frac{f(t)-f(x)}{t-x} \geq 0.$$

Letting $t \to x,$ we see that

$$f'(x) \geq 0. \tag{11}$$

If $x < t < x+\delta,$ then

$$\frac{f(t)-f(x)}{t-x} \leq 0.$$

Letting $t \to x,$ we see that

$$f'(x) \leq 0. \tag{12}$$

Combining (11) and (12), we obtain $f'(x) = 0.$

(2.4) Theorem ***(Generalized Mean Value Theorem)*** If f and g are continuous real functions on $[a,b]$ which are differentiable in (a,b), then there is a point $x \in (a,b)$ at which

$$[f(b)-f(a)]g'(x) = [g(b)-g(a)]f'(x).$$

Note that differentiability is not required at the endpoints.

Proof

Put

$$h(t) = [f(b)-f(a)]g(t) - [g(b)-g(a)]f(t) \qquad (a \leq t \leq b).$$

Then h is continuous on $[a,b]$, h is differentiable in (a,b), and

$$h(a) = f(b)g(a) - f(a)g(b) = h(b). \tag{13}$$

To prove the theorem, we have to show that $h'(x) = 0$ for some $x \in (a,b)$.

Case 1) If h is constant, this holds for every $x \in (a,b)$.

Case 2) If $h(t) > h(a)$ for some $t \in (a,b)$, let x be a point on $[a,b]$ at which h attains its maximum (Ref. Theorem (3.4) of Chapter 3). By (13), $x \in (a,b)$, and Theorem (2.3) shows that $h'(x) = 0.$

Case 3) If $h(t) < h(a)$ for some $t \in (a,b)$, let x be a point on $[a,b]$ at which h attains its minimum (Ref. Theorem (3.4) of Chapter 3). By (13), $x \in (a,b)$, and Theorem (2.3) shows that $h'(x) = 0.$

(2.5) Theorem ***("The" Mean Value Theorem)*** If f is a real continuous function on $[a,b]$ which is differentiable in (a,b), then there is a point $x \in (a,b)$ at which

$$f(b)-f(a) = (b-a)f'(x).$$

Proof

Take $g(x) = x$ in the previous theorem. Then

$$[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x)$$

becomes

$$f(b) - f(a) = (b - a)\, f'(x).$$

(2.6) Theorem Suppose f is differentiable in (a,b).

(a) If $f'(x) \geq 0$ for all $x \in (a,b)$, then f is monotonically increasing.

(b) If $f'(x) = 0$ for all $x \in (a,b)$, then f is constant.

(c) If $f'(x) \leq 0$ for all $x \in (a,b)$, then f is monotonically decreasing.

Proof

Consider the equation

$$f(x_2) - f(x_1) = (x_2 - x_1)\, f'(x), \tag{14}$$

which is valid, for each pair of numbers x_1, x_2 in (a,b), for *some* x between x_1 and x_2.

(*a*) If $f'(x) \geq 0$ for all $x \in (a,b)$, then $x_2 \geq x_1$ implies $x_2 - x_1 \geq 0$ implies $(x_2 - x_1) f'(x) \geq 0$ implies [by (14)], $f(x_2) - f(x_1) \geq 0$ implies $f(x_2) \geq f(x_1)$ implies f is monotonically increasing.

(*b*) If $f'(x) = 0$ then by (14), $f(x_2) - f(x_1) = 0$ for each pair of numbers x_1, x_2 in (a,b), implies $f(x_2) = f(x_1)$ implies f is constant.

(*c*) If $f'(x) \leq 0$ for all $x \in (a,b)$, then $x_2 \geq x_1$ implies $x_2 - x_1 \geq 0$ implies $(x_2 - x_1) f'(x) \leq 0$ implies [by (14)], $f(x_2) - f(x_1) \leq 0$ implies $f(x_2) \leq f(x_1)$ implies f is monotonically decreasing.

3 The Continuity of Derivatives

(3.1) Theorem Suppose f is a real differentiable function on $[a,b]$ and suppose $f'(a) < \lambda < f'(b)$. Then there is a point $x \in (a,b)$ such that $f'(x) = \lambda$.

Proof

Put $g(t) = f(t) - \lambda t.$

Then $g'(a) < 0$, so that $g(t_1) < g(a)$ for some $t_1 \in (a,b)$, and $g'(b) > 0$, so that $g(t_2) < g(b)$ for some $t_2 \in (a,b)$. Hence g attains its minimum on $[a,b]$ (Ref. Theorem

(3.4) of Chapter 3) at some point x such that $a < x < b$. By Theorem (2.3), $g'(x) = 0$. Hence $f'(x) = \lambda$.

(3.2) Theorem Suppose f is a real differentiable function on $[a,b]$ and suppose $f'(a) > \lambda > f'(b)$. Then there is a point $x \in (a,b)$ such that $f'(x) = \lambda$.

Proof Similar to the proof of Theorem (3.1).

(3.3) Corollary If f is differentiable function on $[a,b]$, then f' cannot have any simply discontinuities on $[a,b]$.

But f' may very well have discontinuities of the second kind.

4 L'Hospital's Rule

The limit of the quotient $\frac{f(x)}{g(x)}$ as $x \to a$ can be evaluated using the quotient rule

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}, \quad \text{provided } \lim_{x\to a} g(x) \neq 0.$$

However, the limit problems:

$$\lim_{x\to 0} \frac{\sin x}{x}, \qquad \lim_{x\to 3} \frac{x^2-9}{x^2-x-6}, \qquad \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \qquad \ldots (15)$$

cannot be solved using the quotient rule as the rule requires that the limit of the denominator be different from 0.

It doesn't mean that the limits in (15) do not exist, only that the quotient rule will not determine them.

From the basic analysis course, the reader is familiar with the way of verifying

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Also, the algebraic technique of factoring yields

$$\lim_{x\to 3} \frac{x^2-9}{x^2-x-6} = \lim_{x\to 3} \frac{(x-3)(x+3)}{(x-3)(x+2)} = \lim_{x\to 3} \frac{x+3}{x+2} = \frac{6}{5}.$$

Would it not be nice to have a standard procedure for handling all problems of this type? That is too much to hope for. However, there is a simple rule that works well on a wide

variety of such problems. It is known as l'Hôpital's Rule (pronounced Lö'pëtäl); also called l'Hospital rule or Lhospital rule. The rule was introduced by the French analyst and geometer Guillaume François Antoine de l'Hospital (1661-1704).

***Example* 1** Use l'Hôpital's rule to show that $\lim_{x\to 0}\frac{\sin x}{x}=1$.

Solution

Here limits of both the numerator and denominator is 0. Therefore $\lim_{x\to 0}\frac{\sin x}{x}$ is in the 0/0 form.

Now

$$\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{\cos x}{1}, \text{ using l'Hôpital's Rule and noting that derivative of sin } x \text{ is}$$

$$\cos x \text{ and that of } x \text{ is 1.}$$

$$=\frac{\lim_{x\to 0}\cos x}{\lim_{x\to 0}1}, \quad \text{using quotient rule for limits}$$

$$=\frac{1}{1}=1.$$

***Example* 2** Find $\lim_{x\to\infty}\frac{x}{e^x}$

Solution

Both x and e^x tend to ∞ as $x\to\infty$. Hence limit is in ∞/∞ form.

$$\lim_{x\to\infty}\frac{x}{e^x}=\lim_{x\to\infty}\frac{1}{e^x}, \text{ applying l'Hôpital's Rule}$$

$$= 0.$$

(4.1) Theorem: L'Hospital's Rule Suppose f and g are real and differentiable in (a,b), and $g'(x)\neq 0$ for all $x\in(a,b)$, where $-\infty\le a<b\le+\infty$. Suppose

$$\frac{f'(x)}{g'(x)}\to A \text{ as } x\to a. \tag{16}$$

If

$$f(x)\to 0 \text{ and } g(x)\to 0 \text{ as } x\to a, \tag{17}$$

or if

$$g(x)\to+\infty \text{ as } x\to a, \tag{18}$$

then

$$\frac{f(x)}{g(x)} \to A \quad \text{as } x \to a. \tag{19}$$

The analogous statement is of course also true if $x \to b$, or if $g(x) \to -\infty$ in (18). Let us note that we now use the limit concept in the extended sense of Definition in Section 7 of the previous chapter.

Proof

We first consider the case in which $-\infty \le A < +\infty$. Choose a real number q such that $A < q$, and then choose r such that $A < r < q$. By (16) there is a point $c \in (a,b)$ such that $a < x < c$ implies

$$\frac{f'(x)}{g'(x)} < r. \tag{20}$$

If $a < x < y < c$, then Theorem (1.7) shows that there is a point $t \in (x, y)$ such that

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(t)}{g'(t)} < r. \tag{21}$$

Suppose (17) holds. Letting $x \to a$ in (21), we see that

$$\frac{0 - f(y)}{0 - g(y)} \le r$$

i.e.,
$$\frac{f(y)}{g(y)} \le r < q \qquad (a < y < c). \tag{22}$$

Now suppose that (18) holds. Keeping y fixed in (21), we can choose a point $c_1 \in (a, y)$ such that $g(x) > g(y)$ and $g(x) > 0$ if $a < x < c_1$. $g(x) - g(y) > 0$ and $g(x) > 0$ implies $\frac{g(x) - g(y)}{g(x)} > 0$. Multiplying (21) by $\frac{g(x) - g(y)}{g(x)}$, we obtain

$$\frac{g(x) - g(y)}{g(x)} \cdot \frac{f(x) - f(y)}{g(x) - g(y)} < r \frac{g(x) - g(y)}{g(x)}$$

$$\frac{f(x) - f(y)}{g(x)} < r - r\frac{g(y)}{g(x)}$$

$$\frac{f(x)}{g(x)} < r - r\frac{g(y)}{g(x)} + \frac{f(y)}{g(x)} \qquad (a < x < c_1). \tag{23}$$

If we let $x \to a$ in (23), we get

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} < r - rg(y) \lim_{x \to \infty} \frac{1}{g(x)} + f(y) \lim_{x \to \infty} \frac{1}{g(x)}$$

i.e., $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} < r$, as $\lim_{x\to\infty} \dfrac{1}{g(x)} = 0$, using (18).

Hence there is a point $c_2 \in (a, c_1)$ such that

$$\frac{f(x)}{g(x)} < q \qquad (a < x < c_2). \tag{24}$$

Summing up, (22) and (24) shows that for any q, subject only to the condition $A < q$, there is a point c_2 such that

$$\frac{f(x)}{g(x)} < q \qquad \text{if } a < x < c_2. \tag{25}$$

In the same manner, if $-\infty < A \le +\infty$, and p is chosen so that $p < A$, we can find a point c_3 such that

$$p < \frac{f(x)}{g(x)} \qquad (a < x < c_3), \tag{26}$$

and (19) follows from these two statements (25) and (26).

5 Derivatives of Higher Order

(5.1) Definition If f has a derivative f' on an interval, and if f' is itself differentiable, we denote the derivative of f' by f'' and call f'' the **second derivative** of f. Continuing in this manner, we obtain functions

$$f,\ f',\ f'',\ f^{(3)},\ \ldots,\ f^{(n)},$$

each of which is the derivative of the preceding one. $f^{(n)}$ is called the nth derivative, or the derivative of the order n, of f.

In order for $f^{(n)}(x)$ to exist at a point x, $f^{(n-1)}(t)$ must exist in a neighborhood of x (or in a one-sided neighborhood, if x is an endpoint of the interval on which f is defined), and $f^{(n-1)}$ must be differentiable at x. Since $f^{(n-1)}$ must exist in a neighborhood of x, $f^{(n-2)}$ must be differentiable at that neighborhood.

(5.2) Taylor's Theorem Suppose f is a real function on $[a, b]$, n is a positive integer, $f^{(n-1)}$ is continuous on $[a, b]$, $f^{(n-1)}(t)$ exists for any $t \in (a, b)$. Let α, β be distinct points of $[a, b]$, and define

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k. \tag{27}$$

Then there exists a point x between α and β such that

$$f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!}(\beta-\alpha)^n. \tag{28}$$

(UQ 2006)

Remarks

- For $n = 1$, Taylor's theorem is just the **mean value theorem**: For $n = 1$, (28) becomes

$$f(\beta) = P(\beta) + \frac{f^{(1)}(x)}{1!}(\beta-\alpha)^1 \tag{28A}$$

and from (28) with $n = 1$, we have

$$P(\beta) = \sum_{k=0}^{1-1} \frac{f^{(k)}(\alpha)}{k!}(\beta-\alpha)^k = \frac{f^{(0)}(\alpha)}{0!}(\beta-\alpha)^0 = \frac{f(\alpha)}{1}\times 1 = f(\alpha)$$

Substituting this in (28*A*), we get the mean value theorem

$$f(\beta) = f(\alpha) + f'(x)(\beta-\alpha)$$

i.e., $$f(\beta) - f(\alpha) = f'(x)(\beta-\alpha).$$

- In general, the theorem shows that f can be approximated by a polynomial of degree $n-1$, and that (28) allows us to estimate the error, if we know bounds on $\left|f^{(n)}(x)\right|$.

Proof of Taylors Theorem

Let M be the number defined by

$$f(\beta) = P(\beta) + M(\beta-\alpha)^n \tag{29}$$

and put

$$g(t) = f(t) - P(t) - M(t-\alpha)^n \qquad (a \le t \le b). \tag{30}$$

We have to show that $n!M = f^{(n)}(x)$ for some x between α and β.

Differentiating (30) successively, we get

$$g'(t) = f'(t) - P'(t) - M\frac{d}{dt}(t-\alpha)^n = f'(t) - P'(t) - nM(t-\alpha)^{n-1} \tag{31}$$

$$g''(t) = f''(t) - P''(t) - M\frac{d^2}{dt^2}(t-\alpha)^n = f''(t) - P''(t) - n(n-1)M(t-\alpha)^{n-2} \tag{32}$$

$$\vdots$$

$$g^{(n-1)}(t) = f^{(n-1)}(t) - P^{(n-1)}(t) - M\frac{d^{n-1}}{dt^{n-1}}(t-\alpha)^n$$

$$= f^{(n-1)}(t) - P^{(n-1)}(t) - n(n-1)\cdots[n-(n-2)]M(t-\alpha) \qquad (33)$$

Also, we have

$$g^{(n)}(t) = f^{(n)}(t) - P^{(n)}(t) - M\frac{d^n}{dt^n}(t-\alpha)^n$$

$= f^{(n)}(t) - 0 - Mn!$, since by (27) $P^{(n)}(t)$ is a polynomial of degree $n-1$, and hence

$$P^{(n)}(t) = 0, \text{ and } \frac{d^n}{dt^n}(t-\alpha)^n = n!.$$

i.e., $$g^{(n)}(t) = f^{(n)}(t) - n!M \qquad (a \le t < b). \qquad (34)$$

Hence the proof will be completed if we can show that $g^{(n)}(x) = 0$ for some x between α and β.

Note that for $k = 0, \ldots, n-1$,

$$P^{(k)}(t) = k!\frac{f^{(k)}(\alpha)}{k!} + \text{terms involving } (t-\alpha)$$

implies $$P^{(k)}(\alpha) = f^{(k)}(\alpha) \qquad \text{for } k = 0, \ldots, n-1.$$

From (30), we have

$$g(\alpha) = f(\alpha) - P(\alpha) - M(\alpha-\alpha)^n$$

$$= f(\alpha) - f(\alpha) = 0.$$

From (31) , we have

$$g'(\alpha) = f'(\alpha) - P'(\alpha) - nM(\alpha-\alpha)^{n-1} = f'(\alpha) - P'(\alpha) = f'(\alpha) - f'(\alpha) = 0.$$

From (32) , we have

$$g''(\alpha) = f''(\alpha) - P''(\alpha) - n(n-1)M(\alpha-\alpha)^{n-2} = f''(\alpha) - P''(\alpha) = f''(\alpha) - f''(\alpha) = 0.$$

$$\vdots$$

From (33) , we have

$$g^{(n-1)}(\alpha) = f^{(n-1)}(\alpha) - P^{(n-1)}(\alpha) - n(n-1)\cdots[n-(n-2)]M(\alpha-\alpha)$$

$$= f^{(n-1)}(\alpha) - P^{(n-1)}(\alpha) = f^{(n-1)}(\alpha) - f^{(n-1)}(\alpha) = 0.$$

i.e., $$g(\alpha) = g'(\alpha) = \cdots = g^{(n-1)}(\alpha) = 0.$$

Also, from (30), we have

$$g(\beta) = f(\beta) - P(\beta) - M(\beta - \alpha)^n$$
$$= 0, \text{ using (29).}$$

i.e., we have $g(\alpha) = g(\beta) = 0$ with g continuous and differentiable, hence by mean value theorem, there is some $x_1 \in (\alpha, \beta)$ such that

$$g(\beta) - g(\alpha) = g'(x_1)(\beta - \alpha)$$

i.e., $$0 = g'(x_1)(\beta - \alpha)$$

implies $$g'(x_1) = 0.$$

Now, we have $g'(\alpha) = g'(x_1) = 0$, g' continuous and differentiable, hence by applying mean value theorem for the function g', there is some $x_2 \in (\alpha, x_1)$ such that $g''(x_2) = 0$. After n steps we arrive at the conclusion that $g^{(n)}(x_n) = 0$ for some $x_n \in (\alpha, x_{n-1})$, i.e., for some x_n between α and β.

(5.3) Example Suppose f is defined in a neighborhood of x, and suppose $f''(x)$ exists. Show that $\lim_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x)$. Show by an example that the limit may exist even if $f''(x)$ does not. (UQ 2006)

6 Differentiation of Vector Valued Functions

(6.1) Remarks Definition (1.1) applies without any change to complex functions f defined on $[a, b]$, and Theorems (1.4) and (1.5), as well as their proofs, remain valid. If f_1 and f_2 are the real and imaginary parts of f, that is, if

$$f(t) = f_1(t) + i\, f_2(t)$$

for $a \le t \le b$, where $f_1(t)$ and $f_2(t)$ are real, then we clearly have

$$f'(x) = f_1'(x) + i\, f_2'(x); \tag{35}$$

also, f is differentiable at x if and only if both f_1 and f_2 are differentiable at x.

Passing to vector-valued functions in general, i.e., to functions **f** which map $[a, b]$ into some $\mathbb{R}^k$, we may still apply Definition (1.1) to define $\mathbf{f}'(x)$. The map $\phi(t)$ in (1) is

now, for each t, a point in $\mathbb{R}^k$, and the limit in (2) is taken with respect to the norm of $\mathbb{R}^k$. In other words, $\mathbf{f}'(x)$ is that point of $\mathbb{R}^k$ (if there is one) for which

$$\lim_{t\to x}\left|\frac{\mathbf{f}(t)-\mathbf{f}(x)}{t-x}-\mathbf{f}'(x)\right|=0, \tag{36}$$

and $\mathbf{f}'$ is again a function with values in $\mathbb{R}^k$.

If $f_1, \ldots, f_k$ are the components of **f,** as defined in Theorem (2.9) of the previous chapter, then

$$\mathbf{f}'=\left(f_1', \ldots, f_k'\right), \tag{37}$$

and **f** is differentiable at a point x if and only if each of the functions $f_1, \ldots, f_k$ is differentiable at *x*.

Theorem (1.5) is true in this context as well, and so is Theorem(1.5)(*a*) and (*b*), if fg is replaced by the inner product $\mathbf{f}\cdot\mathbf{g}$.

(6.2) Examples

***Example* 1** **(**Example to show that mean value theorem fails to be true for complex-valued functions).

Define, for real x,

$$f(x)=e^{ix}=\cos x+i\sin x. \tag{38}$$

Then

$$f(2\pi)-f(0)=\cos 2\pi+i\sin 2\pi-\left[\cos 0+i\sin 0\right]=1-1=0, \tag{39}$$

but

$$f'(x)=ie^{ix}, \tag{40}$$

so that $\left|f'(x)\right|=1$ for all real x. Thus mean value theorem fails in this case.

***Example* 2** **(**Example to show that L'Hospitals rule fails to be true for complex-valued functions).

On the segment (0, 1), define $f(x)=x$ and

$$g(x)=x+x^2e^{\frac{i}{x^2}}. \tag{41}$$

$\lim_{x\to 0}\frac{f(x)}{g(x)}$ is in the 0/0 form. Since $\left|e^{it}\right|=1$ for all real t, we see that

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{x}{x + x^2 e^{\frac{i}{x^2}}}$$

$$= \lim_{x\to 0}\frac{1}{1 + xe^{\frac{i}{x^2}}} \text{ (Dividing throughout by } x)$$

$$= 1. \tag{42}$$

Now to evaluate $\lim_{x\to 0}\frac{f'(x)}{g'(x)}$:

$$g'(x) = 1 + \left(2x - \frac{2i}{x}\right)e^{\frac{i}{x^2}} \qquad (0 < x < 1), \tag{43}$$

so that

$$|g'(x)| \geq \left|2x - \frac{2i}{x}\right| - 1 \geq \frac{2}{x} - 1. \tag{44}$$

Hence

$$\left|\frac{f'(x)}{g'(x)}\right| = \frac{1}{|g'(x)|} \leq \frac{x}{2-x} \tag{45}$$

and so

$$\lim_{x\to 0}\frac{f'(x)}{g'(x)} = 0. \tag{46}$$

By (42) and (46), L'Hospital's rule fails in this case. Note also that $g'(x) \neq 0$ on $(0, 1)$, by (44).

However, there is a consequence of the mean value theorem which, for purposes of applications, is almost as useful as Mean Value Theorem (2.5), and which remains true for vector-valued functions: From Theorem (2.5) it follows that

$$|f(b) - f(a)| \leq (b-a)\sup_{a<x<b}|f'(x)|. \tag{47}$$

(6.3) Theorem Suppose $\mathbf{f}$ is a continuous mapping of $[a, b]$ into $\mathbb{R}^k$ and $\mathbf{f}$ is differentiable in (a,b). Then there exists $x \in (a, b)$ such that

$$|\mathbf{f}(b) - \mathbf{f}(a)| \leq (b-a)|\mathbf{f}'(x)|. \tag{UQ 2006}$$

Proof

Put $\mathbf{z} = \mathbf{f}(b) - \mathbf{f}(a)$, and define

$$\varphi(t) = \mathbf{z}\cdot\mathbf{f}(t) \qquad (a \leq t \leq b).$$

Then φ is a real-valued continuous function on $[a, b]$ which is differentiable in (a, b). The mean value theorem shows therefore that

$$\varphi(b)-\varphi(a) = (b-a)\varphi'(x) = (b-a)\mathbf{z}\cdot\mathbf{f}'(x)$$

for some $x \in (a, b)$. On the other hand,

$$\varphi(b)-\varphi(a) = \mathbf{z}\cdot\mathbf{f}(b)-\mathbf{z}\cdot\mathbf{f}(a) = \mathbf{z}\cdot\mathbf{z} = |\mathbf{z}|^2.$$

The Schwarz inequality now gives

$$|\mathbf{z}|^2 = |(b-a)\mathbf{z}\cdot\mathbf{f}'(x)| \le (b-a)|\mathbf{z}||\mathbf{f}'(x)|.$$

Hence $$|\mathbf{z}| \le (b-a)|\mathbf{f}'(x)|,$$

i.e., $$|\mathbf{f}(b)-\mathbf{f}(a)| \le (b-a)|\mathbf{f}'(x)|.$$

Chapter 5

THE RIEMANN-STIELTJES INTEGRAL

1 Definition and Existence of the Integral

(1.1) Definition Let $[a, b]$ be a given interval. By a **partition** P of $[a, b]$ we mean a finite set of points $x_0, x_1, \ldots, x_n$, where

$$a \le x_0 \le x_1 \le \cdots \le x_{n-1} \le x_n = b.$$

We write

$$\Delta x_i = x_i - x_{i-1} \qquad (i = 1, \cdots, n).$$

Now suppose f is a bounded real function defined on $[a, b]$. Corresponding to each partition P of $[a, b]$ we put

$$M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\},$$

$$m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\},$$

$$U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i,$$

$$L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i,$$

and

$$\underline{\int_a^b} f dx = \sup\{L(P, f) : P \text{ is a partition of } [a, b]\}, \tag{1}$$

$$\overline{\int_a^b} f dx = \inf\{U(P, f) : P \text{ is a partition of } [a, b]\}. \tag{2}$$

$\underline{\int_a^b} f dx$ is called **lower Riemann integral** of f over $[a, b]$ and $\overline{\int_a^b} f dx$ is called **upper Riemann integral** of f over $[a, b]$.

(1.2) Definition A bounded function f is said to be **Riemann integrable** on $[a, b]$, if $\underline{\int_a^b} f dx = \overline{\int_a^b} f dx$. This common value (called the **Riemann integral** of f over $[a, b]$) is denoted by

$$\int_a^b f dx$$

or by

$$\int_a^b f(x)dx,$$

or by

$$\Re\int_a^b f(x)dx,$$

(1.3) Notation The set of Riemann integrable functions is denoted by $\Re$.

(1.4) Remark In the above, since f is bounded, there exist two numbers, m and M, such that

$$m \le f(x) \le M \qquad (a \le x \le b).$$

Hence, for every partition P,

$$m(b-a) \le L(P, f) \le U(P, f) \le M(b-a), \qquad \text{(UQ 2008)}$$

so that the numbers $L(P, f)$ and $U(P, f)$ form a bounded set.

Verification:

$$m \le f(x) \le M \qquad (a \le x \le b).$$

implies $\quad m \le m_i \le M_i \le M \qquad (i = 1, 2, \ldots, n).$

implies $\quad m\Delta x_i \le m_i \Delta x_i \le M_i \Delta x_i \le M \Delta x_i \qquad (i = 1, 2, \ldots, n).$

implies $\quad \sum_{i=1}^{n} m\Delta x_i \le \sum_{i=1}^{n} m_i \Delta x_i \le \sum_{i=1}^{n} M_i \Delta x_i \le \sum_{i=1}^{n} M \Delta x_i$

implies $\quad m(b-a) \le L(P, f) \le U(P, f) \le M(b-a).$

This shows that **the upper and lower integrals are defined** for **every** bounded function f. The question of their equality, and hence the question of the integrability of f, is a more delicate one. Instead of investigating it separately for the Riemann integral, we shall immediately consider a more general situation.

(1.5) Definition Let α be a monotonically increasing function on $[a, b]$ (since $\alpha(a)$ and $\alpha(b)$ are finite, it follows that α is bounded on $[a, b]$ (with lower bound $\alpha(a)$ and upper bound $\alpha(b)$). Corresponding to each partition P of $[a, b]$, we write

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}).$$

Since α is monotonically increasing, $\Delta\alpha_i \geq 0$. For any real function f which is bounded on $[a, b]$ we put

$$M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\},$$

$$m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\},$$

$$U(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i,$$

$$L(P, f, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i,$$

and

$$\underline{\int_a^b} f\, d\alpha = \sup\{L(P, f, \alpha) : P \text{ is a partition of } [a, b]\}, \tag{3}$$

$$\overline{\int_a^b} f\, d\alpha = \inf\{U(P, f, \alpha) : P \text{ is a partition of } [a, b]\}. \tag{4}$$

(1.6) Definition In the above, if $\underline{\int_a^b} f\, d\alpha = \overline{\int_a^b} f\, d\alpha,$ we denote their common value by

$$\int_a^b f\, d\alpha \tag{5}$$

or sometimes by

$$\int_a^b f(x) d\alpha(x) \tag{6}$$

and this is the **Riemann-Stieltjes integral** (or simply the **Stieltjes integral**) of f with respect to α, over $[a, b]$.

If (5) exists, we say that f is **integrable with respect to α, in the Riemann sense,** and write $f \in \mathfrak{R}(\alpha)$.

Special Case: By taking $\alpha(x) = x,$ the Riemann integral is seen to be a special case of the Riemann-Stieltjes integral.

(1.7) Definition We say that the partition P^* is a refinement of P if $P^* \supset P$ (i.e., if every point of P is a point of P^*). Given two partitions, P_1 and P_2, we say that P^* is their **common refinement** if $P^* = P_1 \cup P_2$.

(1.8) Theorem If P^* is a refinement of P, then

$$L(P, f, \alpha) \le L(P^*, f, \alpha) \tag{7}$$

and

$$U(P^*, f, \alpha) \le U(P, f, \alpha). \tag{8}$$

Proof

To prove (7), suppose first that P^* contains just one point more than P. Let this extra point be x^*, and suppose $x_{i-1} < x^* < x_i$, where x_{i-1} and x_i are two consecutive points of P. Put

$$w_1 = \inf \left\{ f(x) : x \in \left[x_{i-1}, x^* \right] \right\}$$

$$w_2 = \inf \left\{ f(x) : x \in \left[x^*, x_i \right] \right\}$$

Also, as before $$m_i = \inf \left\{ f(x) : x \in \left[x_{i-1}, x_i \right] \right\}.$$

Clearly $w_1 \ge m_i$ and $w_2 \ge m_i$, and hence

$$\begin{aligned} L(P^*, f, \alpha) - L(P, f, \alpha) \\ &= w_1 \left[\alpha(x^*) - \alpha(x_{i-1}) \right] + w_2 \left[\alpha(x_i) - \alpha(x^*) \right] - m_i \left[\alpha(x_i) - \alpha(x_{i-1}) \right] \\ &= \left(w_1 - m_i \right) \left[\alpha(x^*) - \alpha(x_{i-1}) \right] + \left(w_2 - m_i \right) \left[\alpha(x_i) - \alpha(x^*) \right] \ge 0. \end{aligned}$$

If P^* contains k points more than P, we repeat this reasoning k times, and arrive at (7). The proof of (8) is analogous.

(1.9) Theorem $\underline{\int_a^b} f \, d\alpha \le \bar{\int_a^b} f \, d\alpha.$

Proof

Let P^* be the common refinement of two partitions P_1 and P_2. By Theorem (1.8),

$$L(P_1, f, \alpha) \le L(P^*, f, \alpha) \le U(P^*, f, \alpha) \le U(P_2, f, \alpha).$$

Hence

$$L(P_1, f, \alpha) \le U(P_2, f, \alpha). \tag{9}$$

If P_2 is fixed and the supremum is taken over all P_1, then (9) gives

$$\sup\{L(P_1, f, \alpha) : P_1 \text{ is a partition of } [a, b]\} \le U(P_2, f, \alpha).$$

i.e., $$\int_{_a}^{b} f\, d\alpha \le U(P_2, f, \alpha). \tag{10}$$

Now take the infimum over all P_2, then (10) gives

$$\int_{_a}^{b} f\, d\alpha \le \inf\{U(P_2, f, \alpha) : P_2 \text{ is a partition of } [a, b]\}$$

i.e., $$\int_{_a}^{b} f\, d\alpha \le \bar{\int_a^b} f\, d\alpha.$$

(1.10) Theorem $f \in \Re(\alpha)$ on $[a,b]$ if and only if for every $\varepsilon > 0$ there exists a partition P such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon. \tag{11}$$

(UQ 2006, 2008)

Proof

Assume that for every $\varepsilon > 0$ there exists a partition P such that $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon.$

For every partition P we have

$$L(P, f, \alpha) \le \int_{_a}^{b} f\, d\alpha \le \bar{\int_a^b} f\, d\alpha \le U(P, f, \alpha).$$

Hence

$$0 \le \bar{\int_a^b} f\, d\alpha - \int_{_a}^{b} f\, d\alpha \le U(P, f, \alpha) - L(P, f, \alpha). \tag{12}$$

By assumption, $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon.$ Hence (12) gives

$$0 \le \bar{\int_a^b} f\, d\alpha - \int_{_a}^{b} f\, d\alpha < \varepsilon. \tag{13}$$

Since $\varepsilon > 0$ is arbitrary and can be made very small, (13) gives

$$\bar{\int_a^b} f\, d\alpha - \int_{_a}^{b} f\, d\alpha = 0$$

or $$\bar{\int_a^b} f\, d\alpha = \int_{_a}^{b} f\, d\alpha.$$

i.e., $$f \in \Re(\alpha).$$

Conversely, suppose $f \in \mathfrak{R}(\alpha)$, and let $\varepsilon > 0$ be given. $f \in \mathfrak{R}(\alpha)$ implies $\int_a^b f\,d\alpha = \bar{\int_a^b} f\,d\alpha = \int_{_a}^b f\,d\alpha$. Since

$$\bar{\int_a^b} f\,d\alpha = \inf\{U(P, f, \alpha) : P \text{ is a partition of } [a, b]\},$$

we have $\int_a^b f\,d\alpha + \frac{\varepsilon}{2} = \bar{\int_a^b} f\,d\alpha + \frac{\varepsilon}{2}$ is not a lower bound of the set $\{U(P, f, \alpha) : P \text{ is a partition of } [a, b]\}$ and hence there exist a partition P_2 such that

$$U(P_2, f, \alpha) < \int_a^b f\,d\alpha + \frac{\varepsilon}{2}.$$

Also, since $\int_{_a}^b f\,d\alpha = \sup\{L(P, f, \alpha) : P \text{ is a partition of } [a, b]\}$, we have $\int_a^b f\,d\alpha - \frac{\varepsilon}{2}$ $= \int_{_a}^b f\,d\alpha - \frac{\varepsilon}{2}$ is not an upper bound of the set $\{L(P, f, \alpha) : P \text{ is a partition of } [a, b]\}$ and hence there exist a partition P_1 such that

$$\int_a^b f\,d\alpha - \frac{\varepsilon}{2} < L(P_1, f, \alpha).$$

i.e., corresponding to $\varepsilon > 0$ there exist partitions P_1 and P_2 such that

$$U(P_2, f, \alpha) - \int_a^b f\,d\alpha < \frac{\varepsilon}{2}, \tag{14}$$

$$\int_a^b f\,d\alpha - L(P_1, f, \alpha) < \frac{\varepsilon}{2}. \tag{15}$$

We choose P to be the common refinement of P_1 and P_2. i.e., $P = P_1 \cup P_2$. Then Theorem (1.8), together with (14) and (15), shows that

$$U(P, f, \alpha) \le U(P_2, f, \alpha) < \int_a^b f\,d\alpha + \frac{\varepsilon}{2} < L(P_1, f, \alpha) + \varepsilon \le L(P, f, \alpha) + \varepsilon,$$

so that for this partition P, we have

$$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon.$$

(1.11) Theorem

(a) If

$$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon \tag{16}$$

for some partition P and some ε, then (16) holds (with the same ε) for every refinement of P.

(b) If (16) holds for $P = \{x_0, \ldots, x_n\}$ and if s_i, t_i are arbitrary points in $[x_{i-1}, x_i]$, then

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)| \Delta\alpha_i < \varepsilon.$$

(c) If $f \in \mathfrak{R}(\alpha)$ and the hypotheses of (*b*) hold, then

$$\left| \sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f \, d\alpha \right| < \varepsilon.$$

Proof

(*a*) By Theorem (1.8), if P^* is a refinement of P, then

$$L(P, f, \alpha) \le L(P^*, f, \alpha)$$

and $$U(P^*, f, \alpha) \le U(P, f, \alpha).$$

Hence $$U(P^*, f, \alpha) - L(P^*, f, \alpha) < U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$$ and (*a*) is proved.

(*b*) Suppose (16) holds for $P = \{x_0, \ldots, x_n\}$ and s_i, t_i are arbitrary points in $[x_{i-1}, x_i]$. Then, since $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$, $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$, we have $m_i \le f(s_i)$ and $f(t_i) \le M_i$ i.e., both $f(s_i)$ and $f(t_i)$ lie in $[m_i, M_i]$, so that $|f(s_i) - f(t_i)| \le M_i - m_i$.

Thus

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)| \Delta\alpha_i \le U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon.$$

Hence (*b*) is proved.

(*c*) $f \in \mathfrak{R}(\alpha)$ implies that for every $\varepsilon > 0$ there exists a partition P such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon.$$

From the discussion of (*b*), we have $m_i \le f(s_i) \le M_i$ and hence

$$L(P, f, \alpha) \le \sum_{i=1}^{n} f(s_i)\Delta\alpha_i \le U(P, f, \alpha) \tag{17}$$

Also, we have

$$L(P, f, \alpha) \le \int f \, d\alpha \le U(P, f, \alpha) \tag{18}$$

From (17) and (18), we obtain

$$\left|\sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f\, d\alpha\right| < U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon.$$

(1.12) Theorem If f is continuous on $[a, b]$ then $f \in \Re(\alpha)$ on $[a, b]$.

Proof

$[a, b]$, being a closed and bounded set, is compact. f is continuous on the compact set $[a, b]$ and hence f is uniformly continuous on $[a, b]$. Let $\varepsilon > 0$ be given. Choose $\eta > 0$ so that

$$\left[\alpha(b) - \alpha(a)\right]\eta < \varepsilon.$$

Since f is uniformly continuous on $[a, b]$, corresponding to the $\eta > 0$ there exists a $\delta > 0$ such that

$$\left|f(x) - f(t)\right| < \eta \tag{19}$$

if $x \in [a, b]$, $t \in [a, b]$, and $\left|x - t\right| < \delta$.

If P is any partition on $[a, b]$ such that $\Delta x_i < \delta$ for all i, then (19) implies that

$$M_i - m_i < \eta \qquad (i = 1, \ldots, n) \tag{20}$$

and therefore

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^{n}(M_i - m_i)\Delta\alpha_i \le \eta\sum_{i=1}^{n}\Delta\alpha_i \le \eta\left[\alpha(b) - \alpha(a)\right] < \varepsilon.$$

Hence by Theorem (1.10), $f \in \Re(\alpha)$.

Recall from Chapter 4: Theorem (*Intermediate value theorem*) Let f be a continuous real function on the interval $[a,b]$. If $f(a) < f(b)$ and if c is a number such that $f(a) < c < f(b)$, then there exists a point $x \in (a, b)$ such that $f(x) = c$.

(1.13) Theorem If f is monotonic on $[a, b]$, and if α is continuous on $[a, b]$, then $f \in \Re(\alpha)$. (We still assume, of course, that α is monotonic).

Proof

Let $\varepsilon > 0$ be given. For any positive integer n, choose a partition such that

$$\Delta\alpha_i = \frac{\alpha(b) - \alpha(a)}{n} \qquad (i = 1, \ldots, n).$$

This is possible, by intermediate value theorem, since α is continuous.

We suppose that f is monotonically increasing (the proof is analogous in the other case). Then

$$M_i = f(x_i), \quad m_i = f(x_{i-1}) \qquad (i = 1, \ldots, n),$$

so that

$$U(P, f, \alpha) - L(P, f, \alpha) = \frac{\alpha(b) - \alpha(a)}{n} \sum_{i=1}^{n} \left[f(x_i) - f(x_{i-1}) \right]$$

$$= \frac{\alpha(b) - \alpha(a)}{n} \left[f(b) - f(a) \right] < \varepsilon$$

if n is taken large enough. Hence, by Theorem (1.10), $f \in \Re(\alpha)$.

(1.14) Theorem Suppose f is bounded on $[a, b]$, f has only finitely many points of discontinuities on $[a, b]$, and α is continuous at every point at which f is discontinuous. Then $f \in \Re(\alpha)$.

Proof

Let $\varepsilon > 0$ be given. Put

$$M = \sup \left\{ \left| f(x) \right| : x \in [a, b] \right\}.$$

Let E be the set of points at which f is discontinuous. Since f has only finitely many points of discontinuities, E is finite. Also, α is continuous at every point of E. Since E is finite we can cover E by finitely many disjoint intervals $[u_j, v_j] \subset [a, b]$ such that the sum of the corresponding differences $\alpha(v_j) - \alpha(u_j)$ is less than ε. Furthermore, we can place these intervals in such a way that every point of $E \cap (a, b)$ lies in the interior of some $[u_j, v_j]$.

Remove the segments (u_j, v_j) from $[a, b]$. The remaining set K (being closed and bounded) is compact. Hence f is uniformly continuous on K, and hence corresponding to the given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left| f(s) - f(t) \right| < \varepsilon \quad \text{if} \quad s \in K, t \in K, \left| s - t \right| < \delta.$$

Now form a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ as follows:

Each u_j occurs in P. Each v_j occurs in P. No point of any segment (u_j, v_j) occurs in P. If x_{i-1} is not one of the u_j, then $\Delta x_i < \delta$.

Note that $M_i - m_i \le 2M$ for every i, and $M_i - m_i \le \varepsilon$ unless x_{i-1} is one of the u_j. Hence as in the proof of Theorem (1.12),

$$U(P, f, \alpha) - L(P, f, \alpha) \le \left[\alpha(b) - \alpha(a) \right] \varepsilon + 2M\varepsilon.$$

Since ε is arbitrary, by Theorem (1.10), $f \in \Re(\alpha)$.

(1.15) Remark In the above Theorem, if f and α have a common point of discontinuity, then f need not be in $\Re(\alpha)$.

(1.16) Theorem Suppose $f \in \Re(\alpha)$ on $[a, b]$, $m \le f \le M$, ϕ is continuous on $[m, M]$, and $h(x) = \phi(f(x))$ on $[a, b]$. Then $h \in \Re(\alpha)$ on $[a, b]$.

Proof

Choose $\varepsilon > 0$. Since is continuous on the compact set $[m, M]$, it is uniformly continuous on $[m, M]$, hence there exists $\delta > 0$ such that $\delta < \varepsilon$ and $|\phi(s) - \phi(t)| < \varepsilon$ if $|s - t| \le \delta$ and $s, t \in [m, M]$.

Since $f \in \Re(\alpha)$, there is a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \delta^2. \tag{21}$$

Let

$$M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\},$$

$$m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\},$$

$$M_i^* = \sup\{h(x) : x \in [x_{i-1}, x_i]\},$$

$$m_i^* = \inf\{h(x) : x \in [x_{i-1}, x_i]\}.$$

Divide the numbers $1, \ldots, n$ into two classes: $i \in A$ if $M_i - m_i < \delta$ and $i \in B$ if $M_i - m_i \ge \delta$.

For $i \in A$, our choice of δ shows that $M_i^* - m_i^* \le \varepsilon$.

For $i \in B$, $M_i^* - m_i^* \le 2K$,

where $K = \sup\{|\phi(t)| : m \le t \le M\}$.

On $i \in B$ $\delta \le M_i - m_i$ implies

$$\delta \sum_{i \in B} \Delta\alpha_i \le \sum_{i \in B} (M_i - m_i) \Delta\alpha_i \tag{22}$$

and by (21), (22) becomes

$$\delta \sum_{i \in B} \Delta\alpha_i \le \sum_{i \in B} (M_i - m_i) \Delta\alpha_i < \delta^2 \tag{23}$$

so that $\sum_{i \in B} \Delta\alpha_i < \delta$. It follows that

$$U(P,h,\alpha)-L(P,h,\alpha)=\sum_{i\in A}\left(M_i^*-m_i^*\right)\Delta\alpha_i+\sum_{i\in B}\left(M_i^*-m_i^*\right)\Delta\alpha_i$$

$$\le\varepsilon\left[\alpha(b)-\alpha(a)\right]+2K\delta<\varepsilon\left[\alpha(b)-\alpha(a)+2K\right].$$

Since ε was arbitrary, by Theorem (1.10), $f\in\Re(\alpha)$.

2 Properties of the Integral

(2.1) Theorem

(a) If $f_1\in\Re(\alpha)$ and $f_2\in\Re(\alpha)$ on $[a, b]$, then

$$f_1+f_2\in\Re(\alpha),$$

$$cf\in\Re(\alpha) \text{ for every constant } c, \text{ and}$$

$$\int_a^b\left(f_1+f_2\right)d\alpha=\int_a^b f_1d\alpha+\int_a^b f_2d\alpha,$$

$$\int_a^b cfd\alpha=c\int_a^b f\,d\alpha.$$

(b) If $f_1(x)\le f_2(x)$ on $[a, b]$, then

$$\int_a^b f_1d\alpha\le\int_a^b f_2\,d\alpha.$$

(c) If $f\in\Re(\alpha)$ on $[a, b]$ and if $a<c<b$, then $f\in\Re(\alpha)$ on $[a, c]$ and on $[c, b]$, and

$$\int_a^c f\,d\alpha+\int_c^b f\,d\alpha=\int_a^b f\,d\alpha.$$

(d) If $f\in\Re(\alpha)$ on $[a, b]$ and if $\left|f(x)\right|\le M$ on $[a, b]$, then

$$\left|\int_a^b f\,d\alpha\right|\le M\left[\alpha(b)-\alpha(a)\right].$$

(e) If $f\in\Re(\alpha_1)$ and $f\in\Re(\alpha_2)$, then $f\in\Re(\alpha_1+\alpha_2)$ and

$$\int_a^b f\,d\left(\alpha_1+\alpha_2\right)=\int_a^b f\,d\alpha_1+\int_a^b f\,d\alpha_2;$$

if $f\in\Re(\alpha)$ and c is a positive constant, then $f\in\Re(c\alpha)$ and

$$\int_a^b f\,d\left(c\alpha\right)=c\int_c^b f\,d\alpha.$$

Proof

(*a*) If $f = f_1 + f_2$ and P is any partition of $[a, b]$, we have

$$L(P, f_1, \alpha) + L(P, f_2, \alpha) \le L(P, f, \alpha) \le U(P, f, \alpha) \le U(P, f_1, \alpha) + U(P, f_2, \alpha) \qquad (24)$$

If $f_1 \in \Re(\alpha)$ and $f_2 \in \Re(\alpha)$, let $\varepsilon > 0$ be given. There are partitions P_1 and P_2 such that

$$U(P_1, f_1, \alpha) - L(P_1, f_1, \alpha) < \varepsilon$$

and

$$U(P_2, f_2, \alpha) - L(P_2, f_2, \alpha) < \varepsilon.$$

These inequalities persist if P_1 and P_2 are replaced by their common refinement P. Then (24) implies

$$L(P, f, \alpha) - U(P, f, \alpha) \le U(P_1, f_1, \alpha) + U(P_2, f_2, \alpha) - L(P_1, f_1, \alpha) - L(P_2, f_2, \alpha) < 2\varepsilon,$$

hence by Theorem (1.10), $f \in \Re(\alpha)$. i.e., $f_1 + f_2 \in \Re(\alpha)$.

With the same P we have

$$U(P, f_1, \alpha) < \int f_1 d\alpha + \varepsilon$$

and

$$U(P, f_2, \alpha) < \int f_2 d\alpha + \varepsilon;$$

hence (24) implies

$$\int f\, d\alpha \le U(P, f, \alpha) \le U(P, f_1, \alpha) + U(P, f_2, \alpha) < \int f_1 d\alpha + \int f_2 d\alpha + 2\varepsilon.$$

Since ε was arbitrary, we conclude that

$$\int f\, d\alpha \le \int f_1 d\alpha + \int f_2 d\alpha. \qquad (25)$$

If we replace f_1 and f_2 in (25) by $-f_1$ and $-f_2$, the inequality is reversed, and the equality is proved.

The proofs of the other assertions of Theorem (2.1) are so similar that we omit the details. In part (*c*) the point is that we may restrict ourselves to partitions which contain the point c, in approximating $\int f\, d\alpha$.

(2.2) Theorem If $f \in \Re(\alpha)$ and $g \in \Re(\alpha)$ on $[a, b]$, then

(a) $fg \in \Re(\alpha)$;

(b) $|f| \in \Re(\alpha)$ and $\left| \int_a^b f\, d\alpha \right| \le \int_a^b |f|\, d\alpha.$

Proof

(*a*) Take $\phi(t) = t^2$. Being a polynomial, ϕ is continuous. Also, given $f \in \Re(\alpha)$. By Theorem (1.16) if $h(t) = \phi(f(t))$, then $h \in \Re(\alpha)$. Note that $h(t) = \phi(f(t)) = f^2(t)$ and hence $f^2 \in \Re(\alpha)$.

Also, note the identity

$$4fg = (f+g)^2 - (f-g)^2.$$

$f \in \Re(\alpha)$ and $g \in \Re(\alpha)$ implies by Theorem (2.1)(*a*) that $f+g \in \Re(\alpha)$, and $f-g \in \Re(\alpha)$. Also by the discussion just above, $(f+g)^2 \in \Re(\alpha)$ and $(f-g)^2 \in \Re(\alpha)$. Again, by Theorem (2.1)(*a*) , $(f+g)^2 + (f-g)^2 \in \Re(\alpha)$ and so $4fg \in \Re(\alpha)$ and hence $fg \in \Re(\alpha)$.

(*b*) Take $\phi(t) = |t|$. Then ϕ is continuous. Also, given $f \in \Re(\alpha)$. By Theorem (1.16) if $h(t) = \phi(f(t))$, then $h \in \Re(\alpha)$. Note that $h(t) = \phi(f(t)) = |f|(t)$ and hence $|f| \in \Re(\alpha)$. Choose $c = \pm 1$, so that

$$c\int f\, d\alpha \geq 0.$$

Then

$$\left|\int f\, d\alpha\right| = c\int f\, d\alpha = \int cf\, d\alpha \leq \int |f|\, d\alpha,$$

since $cf \leq |f|$.

(2.3) Theorem The **unit step function** I is defined by

$$I(x) = \begin{cases} 0 & (x \leq 0), \\ 1 & (x > 0). \end{cases}$$

(2.4) Theorem If $a < s < b$, f is bounded on $[a, b]$, f is continuous at s, and $\alpha(x) = I(x-s)$, then

$$\int_a^b f\, d\alpha = f(s).$$

Proof

Consider partitions $P = \{x_0, x_1, x_2, x_3\}$, where $x_0 = a$, and $x_1 = s < x_2 < x_3 = b$. Then

As usual, we take $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$,

$$m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\},$$

$$U(P, f, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i = M_1[\alpha(x_1) - \alpha(x_0)] + M_2[\alpha(x_2) - \alpha(x_1)] + M_3[\alpha(x_3) - \alpha(x_2)]$$

$$= M_1[0-0] + M_2[1-0] + M_3[1-1] = M_2;$$

and $U(P, f, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i = m_1[\alpha(x_1) - \alpha(x_0)] + m_2[\alpha(x_2) - \alpha(x_1)] + m_3[\alpha(x_3) - \alpha(x_2)]$

$$= m_1[0-0] + m_2[1-0] + m_3[1-1] = m_2.$$

Since f is continuous at s, we have $\lim_{x \to s} f(x) = f(s)$ and hence we see that M_2 and m_2 converge to $f(s)$ as $x_2 \to s$.

Hence $$\int_{_a}^{b} f\, d\alpha = \sup\{L(P, f, \alpha) : P \text{ is a partition of } [a, b]\} = f(s)$$

$$\bar{\int_a^b} f\, d\alpha = \inf\{U(P, f, \alpha) : P \text{ is a partition of } [a, b]\} = f(s).$$

Hence $\int_{_a}^{b} f\, d\alpha = = f(s) = \bar{\int_a^b} f\, d\alpha$ and the common value is the Riemann integral $\int_a^b f\, d\alpha = f(s)$.

(2.5) Theorem Suppose $c_n \geq 0$ for $1, 2, 3, \ldots$, $\sum c_n$ converges, $\{s_n\}$ is a sequence of distinct points in (a, b), and

$$\alpha(x) = \sum_{n=1}^{\infty} c_n I(x - s_n). \tag{26}$$

Let f be continuous on $[a, b]$. Then

$$\int_a^b f\, d\alpha = \sum_{n=1}^{\infty} c_n f(s_n). \tag{27}$$

Proof

The comparison test shows that the series (26) converges for every x. Its sum $\alpha(x)$ is evidently monotonic, and $\alpha(a) = 0$, $\alpha(b) = \sum c_n$.

Let $\varepsilon > 0$ be given, and choose N so that

$$\sum_{N+1}^{\infty} c_n < \varepsilon.$$

Put

$$\alpha_1(x) = \sum_{n=1}^{N} c_n I(x - s_n), \qquad \alpha_2(x) = \sum_{N+1}^{\infty} c_n I(x - s_n).$$

$$\int_a^b f\, d\alpha_1 = \int_a^b f\, d\left(\sum_{n=1}^{N} c_n I(x - s_n) \right)$$

$$= \sum_{n=1}^{N} c_n \int_a^b f \, d\left(I(x - s_n)\right), \text{ by Theorem (2.1)}(e)$$

$$= \sum_{n=1}^{N} c_n f(s_n), \text{ by Theorem (2.4).}$$

i.e., $$\int_a^b f \, d\alpha_1 = \sum_{n=1}^{N} c_n f(s_n). \tag{28}$$

Since $\alpha_2(b) - \alpha_2(a) < \varepsilon$,

$$\left| \int_a^b f \, d\alpha_2 \right| \le M\varepsilon, \tag{29}$$

where we take $M = \left\{ \sup \left| f(x) \right| : x \in [a,b] \right\}$.

Since $\alpha = \alpha_1 + \alpha_2$, it follows from Theorem (2.1)(*e*) that $\int_a^b f \, d\alpha = \int_a^b f \, d\alpha_1 + \int_a^b f \, d\alpha_2$. Hence it follows from (29) that

$$\left| \int_a^b f \, d\alpha - \int_a^b f \, d\alpha_1 \right| \le M\varepsilon,$$

and it follows from (28) that

$$\left| \int_a^b f \, d\alpha - \sum_{n=1}^{N} c_n f(s_n) \right| \le M\varepsilon. \tag{30}$$

If we let $N \to \infty$, we obtain $\int_a^b f \, d\alpha = \sum_{n=1}^{\infty} c_n f(s_n)$.

(2.6) Theorem Assume α increases monotonically and $\alpha' \in \mathfrak{R}$ (i.e., α' is Riemann integrable, not Riemann-Steiltjes integrable). Let f be a bounded real function on $[a,b]$. Then $f \in \mathfrak{R}(\alpha)$ if and only if $f\alpha' \in \mathfrak{R}$. In that case

$$\int_a^b f \, d\alpha = \int_a^b f(x)\,\alpha'(x)\,dx. \tag{31}$$

(UQ 2006)

Proof

Let $\varepsilon > 0$ be given and apply Theorem (1.10) to α'. Then there is a partition $P = \{x_0, \ldots, x_n\}$ of $[a,b]$ such that

$$U(P, \alpha') - L(P, \alpha') < \varepsilon. \tag{32}$$

The mean value theorem furnishes points $t_i \in [x_{i-1}, x_i]$ such that

$$\Delta\alpha_i = \alpha'(t_i)\Delta x_i$$

for $i = 1, \ldots, n$. If $s_i \in [x_{i-1}, x_i]$, then

$$\sum_{i=1}^{n} |\alpha'(s_i) - \alpha'(t_i)|\,\Delta x_i < \varepsilon, \tag{33}$$

by (32) and Theorem (1.11)(*c*). Put $M = \{\sup |f(x)| : x \in [a,b]\}$. Since

$$\sum_{i=1}^{n} f(s_i)\Delta\alpha_i = \sum_{i=1}^{n} f(s_i)\alpha'(t_i)\Delta x_i$$

it follows from (33) that

$$\left|\sum_{i=1}^{n} f(s_i)\Delta\alpha_i - \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i\right| = \left|\sum_{i=1}^{n} f(s_i)\alpha'(t_i)\Delta x_i - \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i\right|$$

$$\leq \left|\sum_{i=1}^{n} M\alpha'(t_i)\Delta x_i - \sum_{i=1}^{n} M\alpha'(s_i)\Delta x_i\right| \leq M\sum_{i=1}^{n} |\alpha'(s_i) - \alpha'(t_i)|\,\Delta x_i < M\varepsilon$$

i.e.,
$$\left|\sum_{i=1}^{n} f(s_i)\Delta\alpha_i - \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i\right| \leq M\varepsilon. \tag{34}$$

In particular,

$$\sum_{i=1}^{n} f(s_i)\Delta\alpha_i \leq U(P, f\alpha') + M\varepsilon,$$

for all choices of $s_i \in [x_{i-1},\ x_i]$, so that

$$U(P, f, \alpha) \leq U(P, f\alpha') + M\varepsilon.$$

The same argument leads from (34) to

$$U(P, f\alpha') \leq U(P, f, \alpha) + M\varepsilon.$$

Thus

$$|U(P, f, \alpha) - U(P, f\alpha')| \leq M\varepsilon. \tag{35}$$

Now note that (32) remains true if P is replaced by any refinement. Hence (35) also remains true. We conclude that

$$\left|\bar{\int_a^b} f\,d\alpha - \bar{\int_a^b} f(x)\alpha'(x)\,dx\right| \leq M\varepsilon.$$

But ε is arbitrary. Hence

$$\bar{\int_a^b} f\,d\alpha = \bar{\int_a^b} f(x)\alpha'(x)\,dx, \tag{36}$$

for *any* bounded f. The equality of the lower integrals follows from (34) in exactly the same way. This completes the proof of the theorem.

(2.7) Example Evaluate $\int_0^{\frac{\pi}{2}} \cos x \, d(\sin x)$.

Solution Take $\alpha = \sin x$. Then α increases monotonically in 0 to $\frac{\pi}{2}$. Also, $\cos x$ is bounded on $\left[0, \frac{\pi}{2}\right]$. Hence by Theorem (2.6),

$$\int_0^{\frac{\pi}{2}} \cos x \, d(\sin x) = \int_a^b f \, d\alpha = \int_a^b f(x)\alpha'(x)dx = \int_0^{\frac{\pi}{2}} \cos x \cos x \, dx = \int_0^{\frac{\pi}{2}} \cos^2 x \, dx$$

$$= \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}, \text{ by the reduction formula}$$

$$\int_0^{\pi/2} \cos^n x \, dx = \begin{cases} \frac{n-1}{n} \frac{n-3}{n-2} \frac{n-5}{n-4} \cdots \frac{2}{3} .1, & \text{when } n \text{ is odd.} \\ \frac{n-1}{n} \frac{n-3}{n-2} \frac{n-5}{n-4} \cdots \frac{1}{2} \frac{\pi}{2}, & \text{when } n \text{ is even.} \end{cases}$$

(2.8) Example Evaluate $\int_0^1 x \, d\left([x] - x\right)$ where $[x]$ denotes the largest integer not greater than x. (UQ 2003)

Solution Take $\alpha = [x] - x$. Then α decreases monotonically in 0 to 1 and $-\alpha = x - [x]$ increases monotonically in 0 to 1. Also, x is bounded on $[0, 1]$. Hence by Theorem (2.6),

$$\int_0^1 x \, d\left([x] - x\right) = -\int_0^1 x \, d\left(x - [x]\right) = -\int_a^b f \, d\beta = \int_a^b f(x)\beta'(x)dx = -\int_0^1 x \cdot 1 \cdot dx, \text{ as } \beta'(x) = \frac{d}{dx}(x - 0) = 1.$$

$$= -\left[\frac{x^2}{2}\right]_0^1 = -\frac{1}{2}.$$

(2.9) Theorem (change of variable) Suppose φ is a strictly increasing continuous function that maps an interval $[A, B]$ onto $[a, b]$. Suppose α is monotonically increasing on $[a, b]$ and $f \in \Re(\alpha)$ on $[a, b]$. Define β and g on $[A, B]$ by

$$\beta(y) = \alpha(\varphi(y)), \qquad g(y) = f(\varphi(y)). \tag{37}$$

Then $g \in \Re(\beta)$ and

$$\int_A^B g\, d\beta = \int_a^b f\, d\alpha. \tag{38}$$

Proof

To each partition $P = \{x_0, \ldots, x_n\}$ of $[a,b]$ corresponds a partition $Q = \{y_0, \ldots, y_n\}$ of $[A,B]$, so that $x_i = \varphi(y_i)$. All partitions of $[A,B]$ are obtained in this way. Since the values taken by f on $[x_{i-1}, x_i]$ are exactly the same as those taken by g on $[y_{i-1}, y_i]$, we see that

$$U(Q,g,\beta) = U(P,f,\alpha), \qquad L(Q,g,\beta) = L(P,f,\alpha). \tag{39}$$

Since $f \in \Re(\alpha)$, P can be chosen so that both $U(P,f,\alpha)$ and $L(P,f,\alpha)$ are close to $\int_a^b f\, d\alpha$. Hence (39), shows that $U(Q,g,\beta)$ and $L(Q,g,\beta)$ are close to $\int_a^b f\, d\alpha$ and this combined with Theorem (1.10), shows that $g \in \Re(\beta)$ and $\int_A^B g\, d\beta = \int_a^b f\, d\alpha$. This completes the proof.

Special Case: Take $\alpha(x) = x$. Then $\beta = \varphi$. Assume $\varphi' \in \Re$ on $[A,B]$. If Theorem (2.6) is applied to the left side of (38), we obtain

$$\int_a^b f(x)\, dx = \int_A^B f\big(\varphi(y)\big)\varphi'(y)\, dy. \tag{40}$$

3 Integration and Differentiation

(3.1) Theorem Let $f \in \Re$ on $[a, b]$. For $a \le x \le b$, put

$$F(x) = \int_a^x f(t)\, dt.$$

Then F is continuous on $[a, b]$; furthermore, if f is continuous at a point x_0 of $[a, b]$, then F is differentiable at x_0, and

$$F'(x_0) = f(x_0).$$

Proof

Since $f \in \Re$, f is bounded. Suppose $|f(t)| \le M$ for $a \le t \le b$. If $a \le x < y \le b$, then

$$|F(y)-F(x)| = \left| \int_a^y f(t)\,dt - \int_a^x f(t)\,dt \right| = \left| \int_x^y f(t)\,dt \right| \le M(y-x),$$

by Theorem (2.1)(*c*) and (*d*). Given $\varepsilon > 0$, we see that

$$|F(y)-F(x)| < \varepsilon,$$

provided that $|y-x| < \dfrac{\varepsilon}{M}$. This proves continuity (and, in fact, uniform continuity) of F.

Now suppose f is continuous at x_0. Given $\varepsilon > 0$, choose $\delta > 0$ such that

$$|f(t) - f(x_0)| < \varepsilon$$

if $|t - x_0| < \delta$, and $a \le t \le b$. Hence, if

$$x_0 - \delta < s \le x_0 \le t < x_0 + \delta \quad \text{and} \quad a \le s < t \le b,$$

we have, by Theorem (2.1)(*d*),

$$\left| \frac{F(t)-F(s)}{t-s} - f(x_0) \right| = \left| \frac{1}{t-s} \int_s^t [f(u) - f(x_0)]\,du \right| < \varepsilon.$$

It follows that $F'(x_0) = f(x_0)$.

(3.2) Fundamental Theorem of Calculus If $f \in \mathfrak{R}$ on $[a, b]$ and if there is a differentiable function F on $[a, b]$ such that $F' = f$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Proof

Let $\varepsilon > 0$ be given. Choose a partition $P = \{x_0, \ldots, x_n\}$ of $[a,b]$ so that $U(P, f) - L(P, f) < \varepsilon$. The mean value theorem furnishes points $t_i \in [x_{i-1}, x_i]$ such that

$$F(x_i) - F(x_{i-1}) = f(t_i)\Delta x_i$$

for $i = 1, \ldots, n$. Thus

$$\sum_{i=1}^{n} f(t_i)\Delta x_i = F(b) - F(a).$$

It now follows from Theorem (1.11)(*c*) that

$$\left| F(b) - F(a) - \int_a^b f(x)dx \right| < \varepsilon.$$

Since this holds for every $\varepsilon > 0$, the proof is complete.

(3.3) Theorem (Integration by Parts) Suppose F and G are differentiable functions on $[a, b]$, $F' = f \in \mathfrak{R}$, and $G' = g \in \mathfrak{R}$. Then

$$\int_a^b F(x)g(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx.$$

Proof

Put $H(x) = F(x)G(x)$ and apply Fundamental Theorem of Calculus [Theorem (3.2)] to H and its derivative. Substituting in Theorem (3.2), we have

$$\int_a^b H'(x)dx = H(b) - H(a).$$

Then, noting that $H'(x) = F(x)g(x) + f(x)G(x)$, we have

$$\int_a^b \{F(x)g(x) + f(x)G(x)\}dx = F(b)G(b) - F(a)G(a)$$

i.e., $$\int_a^b F(x)g(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx.$$

4 Integration of Vector-Valued Functions

(4.1) Definition Let $f_1, \ldots, f_k$ be real functions on $[a, b]$, and let $\mathbf{f} = (f_1, \ldots, f_k)$ be the corresponding mapping of $[a, b]$ into $\mathbb{R}^k$. If α increases monotonically on $[a, b]$, to say that $\mathbf{f} \in \mathfrak{R}(\alpha)$ means that $f_j \in \mathfrak{R}(\alpha)$ for $j = 1, \ldots, k$. If this is the case, we define

$$\int_a^b \mathbf{f}d\alpha = \left(\int_a^b f_1 d\alpha, \ldots, \int_a^b f_k d\alpha \right).$$

In other words, $\int_a^b \mathbf{f}d\alpha$ is the point in $\mathbb{R}^k$ whose jth coordinate is $\int_a^b f_j d\alpha$.

It is clear that parts (*a*), (*c*) and (*e*) of Theorem (2.1) are valid for these vector-valued integrals; we simply apply the earlier results to each coordinate. The same is true of Theorems (2.6), (3.1) and (3.2).

(4.2) Theorem ***(Analogue of Theorem (3.2))*** If $\mathbf{f}$ and $\mathbf{F}$ map $[a, b]$ into $\mathbb{R}^k$, if $\mathbf{f} \in \mathfrak{R}$ on $[a, b]$, and if $\mathbf{F}' = \mathbf{f}$, then

$$\int_a^b \mathbf{f}(t)dt = \mathbf{F}(b) - \mathbf{F}(a).$$

We need the following Theorem (3.5) from Chapter 3 in the proof of the Theorem

(4.3) Theorem Suppose f is a continuous 1-1 mapping of a compact metric space X into a metric space *Y*. Then the inverse mapping f^{-1} defined on Y by

$$f^{-1}(f(x)) = x \qquad (x \in X)$$

is a continuous mapping of Y onto X.

(4.4) Theorem ***(Analogue of Theorem (2.2)(b))*** If $\mathbf{f}$ maps $[a, b]$ into $\mathbb{R}^k$ and if $\mathbf{f} \in \mathfrak{R}(\alpha)$ for some monotonically increasing function α on $[a, b]$, then $|\mathbf{f}| \in \mathfrak{R}(\alpha)$, and

$$\left| \int_a^b \mathbf{f}\, d\alpha \right| \le \int_a^b |\mathbf{f}|\, d\alpha. \tag{41}$$

Proof

If $f_1, \ldots, f_k$ are the components of $\mathbf{f}$, then

$$|\mathbf{f}| = \left(f_1^2 + \ldots + f_k^2 \right)^{1/2}. \tag{42}$$

By Theorem (1.16), each of the functions $f_i^2 \in \mathfrak{R}(\alpha)$; hence so does their sum [For details Refer the proof of the Theorem (2.2)]. Since x^2 (being a polynomial) is a continuous function of x, Theorem (4.3) shows that the square-root function (which is the inverse function of x^2) is continuous on $[0, M]$, for every real M. Note that the function $|\mathbf{f}|$ in (42) is the composition of the square root function and the function $f_1^2 + \ldots + f_k^2 \in \mathfrak{R}(\alpha)$. Hence, if we apply Theorem (1.16) once more, (42) shows that $|\mathbf{f}| \in \mathfrak{R}(\alpha)$.

To prove (41), put $\mathbf{y} = (y_1, \ldots, y_k)$, where $y_j = \int f_j\, d\alpha$. Then we have $\mathbf{y} = \int \mathbf{f}\, d\alpha$, and

$$|\mathbf{y}|^2 = \sum y_i^2 = \sum y_j \int f_j\, d\alpha = \int \left(\sum y_j f_j \right) d\alpha.$$

By the Schwarz inequality,

$$\sum y_j f_j(t) \le |\mathbf{y}| |\mathbf{f}(t)| \qquad (a \le t \le b); \tag{43}$$

hence Theorem (2.1)(*b*) implies

$$|\mathbf{y}|^2 \le |\mathbf{y}| \int |\mathbf{f}|\, d\alpha. \tag{44}$$

If $\mathbf{y} = 0$, (41) is trivial. If $\mathbf{y} \neq 0$, division of (44) by $|\mathbf{y}|$ gives (41).

5 Rectifiable Curves

(5.1) Definition A continuous mapping γ of an interval $[a, b]$ into $\mathbb{R}^k$ is called a **curve** in $\mathbb{R}^k$. To emphasize the parameter interval $[a, b]$, we may also say that γ is a **curve on** $[a, b]$.

- If γ is one-to-one, γ is called an **arc**.
- If $\gamma(a) = \gamma(b)$, γ is said to be a **closed curve.**

(5.2) Remark It should be noted that we define a curve to be a *mapping*, not a point set. Of course, with each curve γ in $\mathbb{R}^k$ there is associated a subset of $\mathbb{R}^k$, namely the range of γ, but different curves may have the same range.

(5.3) Definition We associate to each partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$ and to each curve γ on $[a, b]$ the number

$$\Lambda(P, \gamma) = \sum_{i=1}^{n} |\gamma(x_i) - \gamma(x_{i-1})|.$$

The ith term in this sum is the distance (in $\mathbb{R}^k$) between the points $\gamma(x_{i-1})$ and $\gamma(x_i)$. Hence $\Lambda(P, \gamma)$ is the length of a polygonal path with vertices at $\gamma(x_0), \gamma(x_1), \ldots, \gamma(x_n)$, in this order. As our partition becomes finer and finer, this polygon approaches the range of γ more and more closely. This makes it seen reasonable to define the **length of** γ as

$$\Lambda(\gamma) = \sup\{\Lambda(P, \gamma) : P \text{ is a partition of } [a, b]\}.$$

If $\Lambda(\gamma) < \infty$, we say that γ is **rectifiable.**

In certain cases, $\Lambda(\gamma)$ is given by a Riemann integral.

(5.4) Theorem If γ' is continuous on $[a, b]$, then γ is rectifiable, and

$$\Lambda(\gamma) = \int_a^b |\gamma'(t)|\, dt.$$

Proof

If $a \le x_{i-1} < x_i \le b,$ then

$$|\gamma(x_i) - \gamma(x_{i-1})| = \left| \int_{x_{i-1}}^{x_i} \gamma'(t)dt \right| \le \int_{x_{i-1}}^{x_i} |\gamma'(t)|\, dt.$$

Hence

$$\Lambda(P, \gamma) \le \int_a^b |\gamma'(t)|\, dt. \tag{45}$$

To prove the opposite inequality, let $\varepsilon > 0$ be given. Since γ' is continuous on $[a, b]$, γ' is uniformly continuous on $[a, b]$, and hence there exists $\delta > 0$ such that

$$|\gamma'(s) - \gamma'(t)| < \varepsilon \quad \text{if} \quad |s - t| < \delta.$$

Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$, with $\Delta x_i < \delta \ \forall i.$ If $x_{i-1} \le t \le x_i,$ it follows that

$$|\gamma'(t)| \le |\gamma'(x_i)| + \varepsilon.$$

Hence

$$\begin{aligned}
\int_{x_{i-1}}^{x_i} |\gamma'(t)|dt &\le |\gamma'(x_i)|\Delta x_i + \varepsilon \Delta x_i \\
&= \left| \int_{x_{i-1}}^{x_i} \left[\gamma'(t) + \gamma'(x_i) - \gamma'(t)\right] dt \right| + \varepsilon \Delta x_i \\
&\le \left| \int_{x_{i-1}}^{x_i} \gamma'(t)dt \right| + \left| \int_{x_{i-1}}^{x_i} \left[\gamma'(x_i) - \gamma'(t)\right] dt \right| + \varepsilon \Delta x_i \\
&\le |\gamma(x_i) - \gamma(x_{i-1})| + 2\varepsilon \Delta x_i.
\end{aligned}$$

If we add these inequalities, we obtain

$$\begin{aligned}
\int_a^b |\gamma'(t)|dt &\le \Lambda(P, \gamma) + 2\varepsilon(b - a) \\
&\le \Lambda(\gamma) + 2\varepsilon(b - a), \quad \text{by taking supremum over all partitions } P.
\end{aligned}$$

Since ε was arbitrary,

$$\int_a^b |\gamma'(t)|\, dt \le \Lambda(\gamma).$$

This completes the proof.

Chapter 6

SEQUENCES AND SERIES OF FUNCTIONS

1 Discussion of Main Problem

(1.1) Definition Suppose $\{f_n\}$, $n = 1, 2, 3, \ldots$ is a sequence of functions defined on a set E, and suppose that the sequence of numbers $\{f_n(x)\}$ converges for every $x \in E$. We can then define a function f by

$$f(x) = \lim_{n\to\infty} f_n(x) \qquad (x \in E). \tag{1}$$

Under these circumstances we say that $\{f_n\}$ converges on E and that f is the **limit,** or the **limit function,** of $\{f_n\}$. We shall say that "$\{f_n\}$ **converges** of f **pointwise** on E" if (1) holds.

Similarly, if $\sum f_n(x)$ converges for every $x \in E$, and if we define

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \qquad (x \in E) \tag{2}$$

the function f is called the **sum** of the series $\sum f_n$.

The main problem which arises is to determine whether important properties of functions are preserved under the limit operations (1) and (2). For instance, if the functions f_n are continuous, or differentiable, or integrable, is the same true of the limit function? What are the relations between f_n' and f', say, or between the integrals of f_n and that of f?

Recall that ***f* is continuous at *x*** means

$$\lim_{t\to x} f(t) = f(x). \tag{3}$$

i.e.,

$$\lim_{t\to x} f(t) = f(\lim_{t\to x} t).$$

Hence, to ask whether the limit of a sequence of continuous functions (i.e., $\lim_{n\to\infty} f_n$) is continuous is the same as to ask whether

$$\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t), \tag{4}$$

i.e., whether the order in which limit processes are carried out is immaterial. On the left side of (3), we first let $n \to \infty$, then $t \to x$; on the right side, $t \to x$ first, then $n \to \infty$.

The following examples show that limit processes cannot in general be interchanged without affecting the result.

(1.2) Example For $m = 1, 2, 3, \ldots,$ $n = 1, 2, 3, \ldots,$ let

$$s_{m,n} = \frac{m}{m+n}.$$

Then, for every fixed n,

$$\lim_{m\to\infty} s_{m,n} = \lim_{m\to\infty} \frac{1}{1+\frac{n}{m}} = \frac{1}{1} = 1,$$

so that

$$\lim_{n\to\infty} \lim_{m\to\infty} s_{m,n} = 1. \tag{5}$$

On the other hand, for every fixed m,

$$\lim_{n\to\infty} s_{m,n} = \lim_{n\to\infty} \frac{m}{m+n} = 0,$$

so that

$$\lim_{m\to\infty} \lim_{n\to\infty} s_{m,n} = 0. \tag{6}$$

(1.3) Example ***(Convergent series of continuous functions may have a discontinuous sum)*** Let

$$f_n(x) = \frac{x^2}{\left(1+x^2\right)^n} \qquad \left(x \text{ real; } n = 1, 2, \ldots\right),$$

Then each f_n, being the quotient of polynomials, is continuous.

Consider

$$f(x)=\sum_{n=0}^{\infty} f_n(x)=\sum_{n=0}^{\infty}\frac{x^2}{\left(1+x^2\right)^n}. \tag{7}$$

Since $f_n(0)=0,$ we have $f(0)=0.$

For $x\neq 0,$ $\sum_{n=0}^{\infty}\frac{x^2}{\left(1+x^2\right)^n}=x^2\sum_{n=0}^{\infty}\frac{1}{\left(1+x^2\right)^n}$, a geometric series with common ration $\frac{1}{1+x^2}$.

$$=x^2\frac{1}{1-\frac{1}{1+x^2}}=x^2\frac{1}{\frac{1+x^2-1}{1+x^2}}=x^2\frac{1+x^2}{x^2}=1+x^2.$$

Hence from (7), we have

$$f(x)=\begin{cases}0 & (x=0),\\ 1+x^2 & (x\neq 0),\end{cases} \tag{8}$$

Clearly f is not continuous at $x = 0.$ This example shows that a convergent series of continuous functions may have a discontinuous sum.

(1.4) Example ***(Everywhere discontinuous limit function, which is not Riemann-integrable)***

For $m=1,2,3,\ldots,$ put

$$f_m(x)=\lim_{n\to\infty}\left(\cos m!\pi x\right)^{2n}.$$

When $m!x$ is an integer, $\cos m!\pi x$ is ± 1 implies $\left(\cos m!\pi x\right)^{2n}=1$ and hence $f_m(x)=1.$ For all other values of $x,$ $-1<\cos m!\pi x<1$ implies $\lim_{n\to\infty}\left(\cos m!\pi x\right)^{2n}=0$, using the Theorem : "If $|x|<1,$ then $\lim_{n\to\infty}x^n=0.$"

i.e., $$f_m(x)=\begin{cases}1 & \text{when } m!x \text{ is an integer,}\\ 0 & \text{when } m!x \text{ is not an integer,}\end{cases}$$

For irrational x, $m!x$ is not an integer, and hence from the above $f_m(x) = 0 \ \forall m$; hence $f(x) = 0.$

For rational x, say $x = \frac{p}{q}$, where p and q are integers, we see that $m!x$ is an integer if $m \geq q$, so that $f(x) = \lim_{m\to\infty} f_m(x) = \lim_{m\to\infty} 1 = 1.$

$m!x$ is not an integer, and hence from the above $f_m(x) = 0 \ \forall m$; hence $f(x) = \lim_{m\to\infty} f_m(x) = \lim_{m\to\infty} 0 = 0.$

i.e., $$f(x) = \begin{cases} 0 & \text{when } x \text{ is irrational,} \\ 1 & \text{when } x \text{ is rational,} \end{cases}$$

i.e., $$\lim_{m\to\infty} \lim_{n\to\infty} \left(\cos m!\pi x\right)^{2n} = \begin{cases} 0 & \text{when } x \text{ is irrational,} \\ 1 & \text{when } x \text{ is rational.} \end{cases}$$

We have thus obtained an every where discontinuous limit function, which is not Riemann-integrable.

(1.5) Example Let

$$f_n(x) = \frac{\sin nx}{\sqrt{n}} \qquad (x \text{ real; } n = 1, 2, \ldots), \tag{9}$$

Then

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{\sin nx}{\sqrt{n}} = 0.$$

implies $f'(x) = 0.$

From (9), $f_n'(x) = \frac{n\cos nx}{\sqrt{n}} = \sqrt{n}\cos nx,$

so that $\left\{ f_n' \right\}$ does not converge to f'. For instance,

$$f_n'(0) = \sqrt{n}\cos n0 = \sqrt{n}.$$

Hence $\lim_{n\to\infty} f_n'(0) = \lim_{n\to\infty} \sqrt{n} = \infty,$ whereas $f'(0) = 0.$

i.e., $$\lim_{n\to\infty} f_n'(0) \neq \left(\lim_{n\to\infty} f_n\right)'(0).$$

Hence limit and differentiation cannot in general be interchanged.

(1.6) Example ***(Limit of the integral need not be equal to the integral of the limit, even if both are finite)***

Let

$$f_n(x) = n^2 x(1-x^2)^n \qquad (0 \le x \le 1,\ n = 1, 2, \ldots). \tag{10}$$

For $0 < x \le 1$, we have

$$\lim_{n\to\infty} f_n(x) = 0,$$

using the Theorem : " If $p > 0$ and $\alpha \in \square$, then $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0.$"

Since $f_n(0) = 0,$ we see that $\lim_{n\to\infty} f_n(0) = 0.$ Hence, we have

$$\lim_{n\to\infty} f_n(x) = 0 \qquad (0 \le x \le 1). \tag{11}$$

A simple calculation shows that

$$\int_0^1 x(1-x^2)^n\,dx = \frac{1}{2n+2}.$$ (Hint: Put $x^2 = t$, then $2x\,dx = dt$ and limits of integration are $t = 0$ and $t = 1$.)

Thus, inspite of (11),

$$\int_0^1 f_n(x)\,dx = \frac{n^2}{2n+2} \to +\infty \quad \text{as } n \to \infty.$$

If, in (10), we replace n^2 by n, (11) still holds, but we now have

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \lim_{n\to\infty} \frac{n}{2n+2} = \lim_{n\to\infty} \frac{1}{2+\frac{2}{n}} = \frac{1}{2},$$

whereas $$\int_0^1 \left[\lim_{n\to\infty} f_n(x)\right] dx = \int_0^1 0\,dx = 0.$$

Thus the limit of the integral need not be equal to the integral of the limit, even if both are finite.

In the coming section, we define a new mode of convergence which enables us to interchange the limit processes.

2 Uniform Convergence

(2.1) Definition We say that a sequence of functions $\{f_n\}$, $n=1,2,3,\ldots$ **converges uniformly on** E to a function f if for every $\varepsilon>0$ there is an integer N such that $n\geq N$ implies

$$|f_n(x)-f(x)|\leq\varepsilon \tag{12}$$

for all $x\in E$.

(2.2) Remarks

- Every uniformly convergent sequence of functions is pointwise convergent.
- **The difference between pointwise convergence and uniform convergence:**

  If $\{f_n\}$ converges pointwise on E, then there exists a function f such that , for every $\varepsilon>0$, and for every $x\in E$, there is an integer N, depending on ε and on x, such that (12) holds if $n\geq N$; if $\{f_n\}$ converges uniformly on E, then there exists a function f such that , for every $\varepsilon>0$, there is an integer N, depending only on ε, such that (12) holds if $n\geq N$.

(2.3) Definition We say that the series $\sum f_n(x)$ **converges uniformly on** E if the sequence $\{s_n\}$ of nth partial sums defined by

$$\sum_{i=1}^{n} f_i(x)=s_n(x)$$

converges uniformly on E.

(2.4) Theorem The sequence of functions $\{f_n\}$, defined on E, converges uniformly on E if and only if for every $\varepsilon>0$ there exists an integer N such that $m\geq N$, $n\geq N$, $x\in E$ implies

$$|f_n(x) - f_m(x)| \le \varepsilon . \qquad (13)$$

Proof

Suppose $\{f_n\}$ converges uniformly on E, and let f be the limit function. Then there is an integer N such that $n \ge N$, $x \in E$ implies

$$|f_n(x) - f(x)| \le \frac{\varepsilon}{2},$$

so that

$$|f_n(x) - f_m(x)| = |f_n(x) - f(x) + f(x) - f_m(x)| \le |f_n(x) - f(x)| + |f(x) - f_m(x)| \le \varepsilon$$

if $m \ge N$, $n \ge N$, $x \in E$.

Conversely, suppose the Cauchy condition holds. By Theorem: "In $\mathbb{R}^k$, every Cauchy sequence converges", for every x the sequence $\{f_n(x)\}$ converges, say to $f(x)$. Thus the sequence $\{f_n\}$ converges on E, to f. We have to prove that the convergence is uniform.

Let $\varepsilon > 0$ be given, and choose N such that (13) holds. Fix n, and let $m \to \infty$ in (13). Since $f_m(x) \to f(x)$ as $m \to \infty$, this gives

$$|f_n(x) - f(x)| \le \varepsilon$$

for every $n \ge N$, $x \in E$, which shows that $\{f_n\}$ converges on E, to f uniformly.

(2.5) Theorem Suppose

$$\lim_{n\to\infty} f_n(x) = f(x) \qquad (x \in E).$$

Put

$$M_n = \sup_{x\in E} |f_n(x) - f(x)|..$$

Then $f_n \to f$ uniformly on E if and only if $M_n \to 0$ as $n \to \infty$.

Proof

$f_n \to f$ uniformly on E if and only if for every $\varepsilon > 0$ there is an integer N such that $n \ge N$ implies

$$\left| f_n(x) - f(x) \right| \le \varepsilon \tag{12}$$

for all $x \in E$.

This is if and only if

$$M_n = \sup_{x \in E} \left| f_n(x) - f(x) \right| \le \varepsilon \quad \text{for} \quad n \ge N$$

if and only if $\left| M_n - 0 \right| \le \varepsilon \qquad \text{for} \quad n \ge N$

if and only if (from the definition of limit of sequences) $M_n \to 0$ as $n \to \infty$.

(2.6) Theorem (Weierstrass M (Majorant) Test) Suppose $\{ f_n \}$ is a sequence of functions defined on E, and suppose

$$\left| f_n(x) \right| \le M_n \qquad (x \in E,\ n = 1,\ 2,\ 3,\ \ldots).$$

Then $\sum f_n$ converges uniformly on E if $\sum M_n$ converges.

Proof

If $\sum M_n$ converges, then, its sequence of nth partial sums $\sum_{i=1}^{n} M_i$ converges and hence the sequence $\left\{ \sum_{i=1}^{n} M_i \right\}$ is Cauchy. Hence for $\varepsilon > 0$, with $m > n$ with m and n very large,

$$\left| \sum_{i=1}^{m} M_i - \sum_{i=1}^{n} M_i \right| \le \varepsilon$$

implies

$$\left| \sum_{i=n+1}^{m} M_i \right| \le \varepsilon$$

implies

$$\left| \sum_{i=n+1}^{m} f_i(x) \right| \le \left| \sum_{i=n+1}^{m} M_i \right| \le \varepsilon \qquad (x \in E).$$

i.e., $$\left| \sum_{i=1}^{m} f_i(x) - \sum_{i=1}^{n} f_i(x) \right| \le \varepsilon \qquad (x \in E)$$

when *m* and n very large. Hence by Theorem (2.4) the sequence of functions $\left\{\sum_{i=1}^{n} f_i\right\}$ (i.e., the sequence of *n*th partial sums) is uniformly convergent. Hence, the series $\sum f_n$ converges uniformly on E. This completes the proof.

3 Uniform Convergence and Continuity

(3.1) Theorem Suppose $f_n \to f$ uniformly on a set E in a metric space. Let x be a limit point of *E*, and suppose that

$$\lim_{t\to x} f_n(t) = A_n \qquad (n = 1, 2, 3, \ldots). \tag{13}$$

Then $\{A_n\}$ converges, and

$$\lim_{t\to x} f(t) = \lim_{n\to\infty} A_n. \tag{14}$$

i.e.,
$$\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t). \tag{15}$$

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Proof

Let $\varepsilon > 0$ be given. By the uniform convergence of $\{f_n\}$, by Theorem (2.4), there exists there exists an integer N such that $m \geq N$, $n \geq N$, $t \in E$ implies

$$|f_n(t) - f_m(t)| \leq \varepsilon. \tag{16}$$

Letting $t \to x$ in (16), we obtain

$$|A_n - A_m| \leq \varepsilon$$

for $m \geq N$, $n \geq N$, so that $\{A_n\}$ is a Cauchy sequence and therefore converges, say to *A*.

Next,

$$|f(t) - A| \leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A|. \tag{17}$$

We first choose n such that

$$|f(t) - f_n(t)| \leq \frac{\varepsilon}{3} \tag{18}$$

for all $t \in E$ (the same n for all t is possible by the uniform convergence), and such that

$$|A_n - A| \le \frac{\varepsilon}{3}. \tag{19}$$

Then, for this n, we choose a neighborhood V of x such that

$$|f_n(t) - A_n| \le \frac{\varepsilon}{3} \tag{20}$$

if $t \in V \cap E$, $t \ne x$. [(20) is possible because $\lim_{t \to x} f_n(t) = A_n$].

Substituting the inequalities (18) to (20) into (17), we see that

$$|f(t) - A| \le \varepsilon, \tag{21}$$

provided $t \in V \cap E$, $t \ne x$. This is equivalent to saying that $\lim_{t \to x} f(t) = A$. i.e., $\lim_{t \to x} f(t) = \lim_{n \to \infty} A_n$, as desired.

(3.2) Theorem (Corollary to Theorem (3.1)) Suppose $\{f_n\}$ is a sequence of continuous functions on E, and if $f_n \to f$ uniformly on E, then f is continuous on E.

Proof

$\{f_n\}$ is a sequence of continuous functions on E implies that each f_n is continuous at each point of E. Let x be a point on E. We have to show that f is continuous at x, then since x is arbitrary point, it follows that f is continuous on E.

Since each f_n is continuous at x, we have

$$\lim_{t \to x} f_n(t) = f_n(x) \qquad (n = 1, 2, 3, \ldots).$$

Then, since $f_n \to f$ uniformly on E, using the previous theorem we have

$$\lim_{t \to x} f(t) = \lim_{n \to \infty} f_n(x).$$

i.e.,

$$\lim_{t \to x} f(t) = f(x),$$

showing that f is continuous at x.

The converse of Theorem (3.2) is not true; i.e., a sequence of continuous functions may converge to a continuous function, although the convergence is not uniform.

(3.3) Theorem Suppose K is compact, and

(a) $\{f_n\}$ is a sequence of continuous functions on K,

(b) $\{f_n\}$ converges pointwise to a continuous function f on K,

(c) $f_n(x) \geq f_{n+1}(x)$ for all $x \in K$, $n = 1, 2, 3, \ldots$

Then $f_n \to f$ uniformly on K.

Proof

Put $g_n = f_n - f$. Then g_n is continuous, $g_n \to 0$ pointwise, and $g_n \geq g_{n+1}$. We have to prove that $g_n \to 0$ uniformly on K.

Let $\varepsilon > 0$ be given. Let

$$K_n = \{x \in K \mid g_n(x) \geq \varepsilon\}.$$

Since g_n is continuous, $K_n = g_n^{-1}([\varepsilon, \infty))$ is closed, and hence is compact. Since $f_n \geq f_{n+1}$ we have $g_n \geq g_{n+1}$ and hence $K_n \supset K_{n+1}$. Fix $x \in K$. Since $g_n(x) \to 0$, we see that $x \notin K_n$ if n is sufficiently large. Thus $x \notin \bigcap K_n$. In other words, $\bigcap K_n$ is empty. Hence K_N is empty for some N. It follows that $0 \leq g_n(x) < \varepsilon$ $\forall x \in K$ and $\forall n \geq N$. This proves the theorem.

(3.4) Remark In the above theorem compactness is really needed. For instance, if

$$f_n(x) = \frac{1}{nx+1} \qquad (0 < x < 1;\ n = 1, 2, 3, \ldots)$$

then $f_n(x) \to 0$ monotonically in (0, 1), but the convergence is not uniform.

(3.5) Definition If X is a metric space, $\wp(X)$ will denote the set of all complex valued, continuous, bounded functions with domain X.

We associate with each $f \in \wp(X)$ its **supremum norm**

$$\|f\| = \sup\{|f(x)| : x \in X\}.$$

Since f is assumed to be bounded, by the supremum property of $\square$, $\|f\| < \infty$.

(i) $\|f\| = 0$ if and only if $\sup\{|f(x)| : x \in X\} = 0$ if and only if $|f(x)| = 0$ for all $x \in X$ if and only if $f(x) = 0$ for all $x \in X$ if and only if $f = 0$, the zero function.

(ii) Note that, for all $x \in X$

$$|(f+g)(x)| = |f(x)+g(x)| \le |f(x)| + |g(x)| \le \|f\| + \|g\|.$$

Hence $$\sup\{|(f+g)(x)| : x \in X\} \le \|f\| + \|g\|$$

i.e., $$\|f+g\| \le \|f\| + \|g\|.$$

For $f \in \wp(X)$ and $g \in \wp(X)$ define $d(f,g) = \|f-g\|$. Then d is a metric on $\wp(X)$: For,

(i) $d(f,g) = 0 \Leftrightarrow \|f-g\| = 0 \Leftrightarrow f-g = 0 \Leftrightarrow f = g.$

(ii) $d(f,g) = \|f-g\| = \|-(f-g)\| = \|g-f\| = d(g,f).$

(iii) $d(f,g) = \|f-g\| = \|f-h+h-g\| \le \|f-h\| + \|h-g\| = d(f,h) + d(h,g).$

Thus $\wp(X)$ is a metric space.

Now Theorem (2.5) can be rephrased as follows:

(3.6) Theorem ***(Theorem (2.5))*** A sequence $\{f_n\}$ converges to f with respect to the metric of $\wp(X)$ if and only if $f_n \to f$ uniformly on X.

Closed subsets of $\wp(X)$ are called *uniformly closed,* the closure of a set $\mathcal{A} \subset \wp(X)$ is called its *uniform closure,* and so on.

(3.7) Theorem The above metric makes $\wp(X)$ into a complete metric space.

Proof

We have to show that every Cauchy sequence in $\wp(X)$ is convergent. Let $\{f_n\}$ be a Cauchy sequence in $\wp(X)$. This means that to each $\varepsilon > 0$ there corresponds an N such that

$\|f_n - f_m\| < \varepsilon$ if $n \geq N$ and $m \geq N$.

i.e., $\sup\{|(f_n - f_m)(x)| : x \in X\} < \varepsilon$ if $n \geq N$ and $m \geq N$.

i.e., $|f_n(x) - f_m(x)| < \varepsilon$ if $x \in X$ $n \geq N$ and $m \geq N$.

Hence it follows by Theorem (2.4) that there is a function f with domain X to which $\{f_n\}$ converges uniformly. By Theorem (3.2), since each f_n is continuous, f is continuous. Since there is an n such that $|f(x) - f_n(x)| < 1 \ \forall x \in X$, and hence $|f_n(x)| < 1 + |f(x)| \ \forall x \in X$, so that f_n is bounded. Hence, f is bounded.

Thus $f \in \wp(X)$, and since $f_n \to f$ uniformly on X, we have $\|f - f_n\| \to 0$ as $n \to \infty$.

4 Uniform Convergence and Integration

(4.1) Theorem Let α be monotonically increasing on $[a, b]$. Suppose $f_n \in \Re(\alpha)$ on $[a, b]$, for $n = 1, 2, 3, \ldots$ and suppose $f_n \to f$ uniformly on $[a, b]$. Then $f \in \Re(\alpha)$ on $[a, b]$, and

$$\int_a^b f \, d\alpha = \lim_{n \to \infty} \int_a^b f_n \, d\alpha. \tag{22}$$

(The existence of the limit is part of the conclusion.)

Proof

It suffices to prove this for real f_n. Put

$$\varepsilon_n = \sup |f_n(x) - f(x)|, \tag{23}$$

the supremum being taken over $a \leq x \leq b$. Then

$$f_n - \varepsilon_n \leq f \leq f_n + \varepsilon_n,$$

so that the upper and lower integrals of f satisfy

$$\int_a^b (f_n - \varepsilon_n)\, d\alpha \le \underline{\int} f\, d\alpha \le \overline{\int} f\, d\alpha \le \int_a^b (f_n + \varepsilon_n)\, d\alpha. \qquad (24)$$

Hence

$$0 \le \overline{\int} f\, d\alpha - \underline{\int} f\, d\alpha \le 2\varepsilon_n [\alpha(b) - \alpha(a)]. \qquad (25)$$

Since $f_n \to f$ uniformly on $[a, b]$, using Theorem (2.5) , we have $\varepsilon_n \to 0$ as $n \to \infty$ and hence from (25), the upper and lower integrals of f are equal.

Thus $f \in \Re(\alpha)$. Another application of (24) now yields

$$\left| \int_a^b f\, d\alpha - \int_a^b f_n d\alpha \right| \le \varepsilon_n [\alpha(b) - \alpha(a)]. \qquad (26)$$

Since $f_n \to f$ uniformly on $[a, b]$, using Theorem (2.5) , we have $\varepsilon_n \to 0$ as $n \to \infty$ and hence from (26), we have

$$\int_a^b f\, d\alpha = \lim_{n\to\infty} \int_a^b f_n\, d\alpha.$$

(4.2) Corollary If $f_n \in \Re(\alpha)$ on $[a, b]$, and if

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \qquad (a \le x \le b),$$

the series converging uniformly on $[a, b]$, then

$$\int_a^b f\, d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n\, d\alpha. \qquad (27)$$

In other words, the series may be integrated term by term.

5 Uniform Convergence and Differentiation

(5.1) Theorem Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a, b]$ and such that $\{f_n(x_0)\}$ converges from some point $x_0 \in [a, b]$. If $\{f_n'\}$ converges uniformly on $[a, b]$, then $\{f_n\}$ converges uniformly on $[a, b]$, to a function f, and

$$f'(x) = \lim_{n\to\infty} f_n'(x) \qquad (a \le x \le b). \tag{28}$$

(UQ 2006)

Proof

Let $\varepsilon > 0$ be given. Choose N such that $n \ge N$, $m \ge N$, implies

$$|f_n(x_0) - f_m(x_0)| < \frac{\varepsilon}{2} \tag{29}$$

and

$$|f_n'(t) - f_m'(t)| < \frac{\varepsilon}{2(b-a)} \qquad (a \le t \le b). \tag{30}$$

If we apply the mean value theorem to the function $f_n - f_m$, (30) shows that

$$|f_n(x) - f_m(x) - [f_n(t) - f_m(t)]| \le \frac{|x-t|\varepsilon}{2(b-a)} \le \frac{\varepsilon}{2} \tag{31}$$

for any x and t on $[a, b]$, if $n \ge N$, $m \ge N$. The inequality

$$|f_n(x) - f_m(x)| \le |f_n(x) - f_m(x) - f_n(x_0) + f_m(x_0)| + |f_n(x_0) - f_m(x_0)|$$

implies, by (29) and (31), that

$$|f_n(x) - f_m(x)| < \varepsilon \qquad (a \le x \le b, n \ge N, m \ge N),$$

so that $\{f_n\}$ converges uniformly on $[a, b]$. Let

$$f(x) = \lim_{n\to\infty} f_n(x) \qquad (a \le x \le b).$$

Let us now fix a point $x \in [a, b]$ and define

$$\phi_n(t) = \frac{f_n(t) - f_n(x)}{t - x}, \qquad \phi(t) = \frac{f(t) - f(x)}{t - x} \tag{32}$$

for $a \le t \le b$, $t \ne x$. Then

$$\lim_{t\to x} \phi_n(t) = f_n'(x) \qquad (n = 1, 2, 3, \ldots) \tag{33}$$

The first inequality in (31) shows that

$$|\phi_n(t) - \phi(t)| < \frac{\varepsilon}{2(b-a)} \qquad (n \ge N, \ m \ge N),$$

so that $\{\phi_n\}$ converges uniformly, for $t \neq x$. Since $\{f_n\}$ converges uniformly to f, we conclude from (32) that

$$\lim_{n\to\infty} \phi_n(t) = \phi(t) \tag{34}$$

uniformly for $a \leq t \leq b$, $t \neq x$.

If we now apply Theorem (3.1) to $\{\phi_n\}$, (33) and (34) show that

$$\lim_{t\to x} \phi(t) = \lim_{n\to\infty} f_n'(x);$$

and this is (28), by the definition of $\phi(t)$.

(5.2) Theorem There exists a real continuous function on the real line which is nowhere differentiable.

Proof

Define

$$\varphi(x) = |x| \qquad (-1 \leq x \leq 1) \tag{35}$$

and extend the definition of $\varphi(x)$ to all real x by requiring that

$$\varphi(x+2) = \varphi(x). \tag{36}$$

Then, for all s and t,

$$|\varphi(s) - \varphi(t)| \leq |s - t|. \tag{37}$$

In particular, φ is continuous on $\mathbb{R}$. Define

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \varphi(4^n x). \tag{38}$$

Since $0 \leq \varphi \leq 1$, Theorem (2.6) shows that the series (38) converges uniformly on $\mathbb{R}$. By Theorem (3.2), f is continuous on $\mathbb{R}$.

Now fix a real number x and a positive integer m. Put

$$\delta_m = \pm\frac{1}{2} \cdot 4^{-m} \tag{39}$$

where the sign is so chosen that no integer lies between $4^m x$ and $4^m(x + \delta_m)$. This can be done, since $4^m |\delta_m| = \frac{1}{2}$. Define

$$\gamma_n = \frac{\varphi(4^n(x+\delta_m)) - \varphi(4^n x)}{\delta_m}. \tag{40}$$

When $n > m$, then $4^n \delta_m$ is an even integer, so that $\gamma_n = 0$. When $0 \le n \le m$, (37) implies that $|\gamma_n| \le 4^n$.

Since $|\gamma_m| = 4^m$, we conclude that

$$\left|\frac{f(x+\delta_m) - f(x)}{\delta_m}\right| = \left|\sum_{n=0}^{m} \left(\frac{3}{4}\right)^n \gamma_n\right|$$

$$\ge 3^m - \sum_{n=0}^{m-1} 3^n = \frac{1}{2}\left(3^m + 1\right).$$

As $m \to \infty$, $\delta_m \to 0$. It follows that f is not differentiable at x.

6 Equicontinuous Families of Functions

(6.1) Definition Let $\{f_n\}$ be a sequence of functions defined on a set E.

- We say that $\{f_n\}$ is **point wise bounded** on E if the sequence $\{f_n(x)\}$ is bounded for every $x \in E$, i.e., if there exists a finite-valued function ϕ defined on E such that

$$|f_n(x)| < \phi(x) \qquad (x \in E,\ n = 1, 2, 3, \ldots).$$

- We say that $\{f_n\}$ is **uniformly bounded** on E if there exists a number M such that

$$|f_n(x)| < M \qquad (x \in E,\ n = 1, 2, 3, \ldots).$$

(6.2) Remark If $\{f_n\}$ is pointwise bounded on E and E_1 is a countable subset of E, it is always possible to find a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}(x)\}$ converges for every $x \in E_1$. This can be done by the diagonal process which is used in the proof of (coming) Theorem (6.6).

(6.3) Example

Let

$$f_n(x) = \sin nx \qquad (0 \le x \le 2\pi,\ n = 1, 2, 3, \ldots).$$

Suppose there exists a sequence $\{n_k\}$ such that $\{\sin n_k x\}$ converges, for every $x \in [0, 2\pi]$. In that case we must have

$$\lim_{k \to \infty} (\sin n_k x - \sin n_{k+1} x) = 0 \qquad (0 \le x \le 2\pi);$$

hence

$$\lim_{k\to\infty}\left(\sin n_k x-\sin n_{k+1}x\right)^2=0 \qquad (0\le x\le 2\pi). \tag{41}$$

By Lebesgue's theorem concerning integration of boundedly convergent sequences, (41) implies

$$\lim_{k\to\infty}\int_0^{2\pi}\left(\sin n_k x-\sin n_{k+1}x\right)^2 dx=0. \tag{42}$$

But a simple calculation shows that

$$\int_0^{2\pi}\left(\sin n_k x-\sin n_{k+1}x\right)^2 dx=2\pi,$$

which contradicts (42).

(6.4) Example

Let

$$f_n(x)=\frac{x^2}{x^2+(1-nx)^2} \qquad (0\le x\le 1,\ n=1,2,3,\ldots).$$

Then $|f_n(x)|\le 1,$ so that $\{f_n\}$ is uniformly bounded on [0, 1]. Also

$$\lim_{n\to\infty} f_n(x)=0 \qquad (0\le x\le 1),$$

but

$$f_n\left(\frac{1}{n}\right)=1 \qquad (n=1,2,3,\ldots),$$

so that no subsequence can converge uniformly on [0, 1].

(6.5) Definition

A family $\mathcal{F}$ of complex functions f defined on a set E in a metric space X is said be **equicontinuous** on E if for every $\varepsilon>0$ there exists a $\delta>0$ such that

$$|f(x)-f(y)|<\varepsilon$$

whenever $d(x,y)<\delta,\ x\in E,\ y\in E,$ and $f\in\mathcal{F}$. Hence d denotes the metric of X.

Every member of an equicontinuous family is uniformly continuous.

The sequence in Example (6.4) is not equicontinuous.

(6.6) Theorem If $\{f_n\}$ is a pointwise bounded sequence of complex functions on a countable set E, then $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}(x)\}$ converges for every $x \in E$.

Proof

Let $\{x_i\}$, $i = 1, 2, 3, \ldots$ be the points of E, arranged in a sequence. Since $\{f_n(x_1)\}$ is bounded, there exists a subsequence, which we shall denote by $\{f_{1,k}\}$, such that $\{f_{1,k}(x_1)\}$ converges as $k \to \infty$.

Let us now consider sequences $S_1, S_2, S_3, \ldots$, which we represent by the array

$$\begin{aligned} S_1&: \ f_{1,1} \ \ f_{1,2} \ \ f_{1,3} \ \ f_{1,4} \ \cdots \\ S_2&: \ f_{2,1} \ \ f_{2,2} \ \ f_{2,3} \ \ f_{2,4} \ \cdots \\ S_3&: \ f_{3,1} \ \ f_{3,2} \ \ f_{3,3} \ \ f_{3,4} \ \cdots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

and which have the following properties:

(a) S_n is a subsequence of S_{n-1} for $n = 2, 3, 4, \ldots$

(b) $\{f_{n,k}(x_n)\}$ converges, as $k \to \infty$ (the boundedness of $\{f_n(x_n)\}$ makes it possible to choose S_n in this way);

(c) The order in which the functions appear is the same in each sequence; i.e., if one function precedes another in S_1, they are in the same relation in every S_n, until one or the other is deleted. Hence, when going from one row in the above array to the next below, functions may move to the left but never to the right.

We now go down the diagonal of the array; i.e., we consider the sequence

$$S: \ f_{1,1} \ \ f_{2,2} \ \ f_{3,3} \ \ f_{4,4} \ \cdots$$

By (*c*), the sequence S (except possibly its first $n - 1$ terms) is a subsequence of S_n for $n = 1, 2, 3, \ldots$ Hence (*b*) implies that $\{f_{n,n}(x_i)\}$ converges, as $n \to \infty$, $\forall x_i \in E$.

(6.7) Theorem If K is a compact metric space, if $f_n \in \mathscr{C}(K)$ for $n = 1, 2, 3, \ldots$, and if $\{f_n\}$ converges uniformly on K, then $\{f_n\}$ is equicontinuous on K.

Proof

Let $\varepsilon > 0$ be given. Since $\{f_n\}$ converges uniformly, there is an integer N such that

$n \geq N$ implies

$$|f_n(x) - f(x)| \leq \varepsilon$$

for all $x \in E$. Since $\|f_n - f_N\| = \sup\{f_n(x) - f_N(x) \mid x \in X\}$, we have

$$\|f_n - f_N\| < \varepsilon \qquad (n > N). \tag{43}$$

Since continuous functions are uniformly continuous on compact sets, there is a $\delta > 0$ such that

$$|f_i(x) - f_i(y)| < \varepsilon \tag{44}$$

if $1 \leq i \leq N$ and $d(x, y) < \delta$.

If $n > N$ and $d(x, y) < \delta$, it follows that

$$|f_n(x) - f_n(y)| \leq |f_n(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f_n(y)| < 3\varepsilon. \tag{45}$$

By (44) and (45), we have, <u>for all n</u>

$$|f_n(x) - f_n(y)| < 3\varepsilon$$

whenever $d(x, y) < \delta$, $x \in E$, $y \in E$ and $f_n \in \wp(K)$. Hence [by the Definition (6.5)] $\{f_n\}$ is equicontinuous on *K*.

(6.8) Theorem If K is compact, if $f_n \in \wp(K)$ for $n = 1, 2, 3, \ldots$, and if $\{f_n\}$ is pointwise bounded and equicontinuous on *K*, then

(a) $\{f_n\}$ is uniformly bounded on K,

(b) $\{f_n\}$ contains a uniformly convergent subsequence. (UQ 2006)

Proof

(*a*) Let $\varepsilon > 0$ be given and choose $\delta > 0$, in accordance with Definition (6.5), so that

$$|f_n(x) - f_n(y)| < \varepsilon \tag{46}$$

for all n, provided $d(x, y) < \delta$.

Since K is compact, there are finitely many points $p_1, \ldots, p_r$ in K such that to every $x \in K$ corresponds at least one p_i with $d(x, p_i) < \delta$. Since $\{f_n\}$ is pointwise bounded, there exist $M_i < \infty$ such that $|f_n(p_i)| < M_i$ for all n. If $M = \max(M_1, \ldots, M_r)$, then $|f(x)| < M + \varepsilon$ for every $x \in K$. This proves (*a*).

(*b*) Let E be a countable dense subset of K. Theorem (6.6) shows that $\{f_n\}$ has a subsequence $\{f_{n_i}\}$ such that $\{f_{n_i}(x)\}$ converges for every $x \in E$.

Put $f_{n_i} = g_i$, to simplify the notation. We shall prove that $\{g_i\}$ converges uniformly on K.

Let $\varepsilon > 0$, and pick $\delta > 0$ as in the beginning of the proof. Let

$$V(x,\delta) = \{y \in K \mid d(x,y) < \delta\}.$$

Since E is dense in K, and K is compact, there are finitely many points $x_1, \ldots, x_m$ in E such that

$$K \subset V(x_1,\delta) \cup \cdots \cup V(x_m,\delta). \tag{47}$$

Since $\{g_i(x)\}$ converges for every $x \in E$, there is an integer N such that

$$|g_i(x_s) - g_j(x_s)| < \varepsilon \tag{48}$$

whenever $i \geq N$, $j \geq N$, $1 \leq s \leq m$.

If $x \in K$, (47) shows that $x \in V(x_s,\delta)$ for some s, so that

$$|g_i(x) - g_i(x_s)| < \varepsilon$$

for every i. If $i \geq N$ and $j \geq N$, it follows from (48) that

$$\begin{aligned} |g_i(x) - g_j(x)| &\leq |g_i(x) - g_i(x_s)| + |g_i(x_s) - g_j(x_s)| + |g_j(x_s) - g_j(x)| \\ &< 3\varepsilon. \end{aligned}$$

This completes the proof.

7 The Stone-Weierstrass Theorem

(7.1) Theorem (***Weierstrass theorem)*** If f is a continuous complex function on $[a, b]$, there exists a sequence of polynomials P_n such that

$$\lim_{n\to\infty} P_n(x) = f(x)$$

uniformly on $[a, b]$. If f is real, the P_n may be taken real.

This is the form in which the theorem was originally discovered by Weierstrass.

Proof

We may assume, without loss of generality, that $[a, b] = [0, 1]$. We may also assume that $f(0) = f(1) = 0$. For if the theorem is proved for this case, consider

$$g(x) = f(x) - f(0) - x[f(1) - f(0)] \qquad (0 \le x \le 1).$$

Here $g(0) = g(1) = 0$, and if g can be obtained as the limit of a uniformly convergent sequence of polynomials, it is clear that the same is true for f, since $f - g$ is a polynomial.

Furthermore, we define $f(x)$ to be zero for x outside $[0, 1]$. Then f is uniformly continuous on the whole line.

We put

$$Q_n(x) = c_n(1-x^2)^n \qquad (n = 1, 2, 3, \ldots), \tag{49}$$

where c_n is chosen so that

$$\int_{-1}^{1} Q_n(x)dx = 1 \qquad (n = 1, 2, 3, \ldots), \tag{50}$$

We need some information about the order of magnitude of c_n. Since

$$\int_{-1}^{1} (1-x^2)^n dx = 2\int_{0}^{1} (1-x^2)^n dx \ge 2\int_{0}^{\frac{1}{\sqrt{n}}} (1-x^2)^n dx$$

$$\ge 2\int_{0}^{\frac{1}{\sqrt{n}}} (1-nx^2)dx \quad \dagger$$

$$= \frac{4}{3\sqrt{n}}$$

$$> \frac{1}{\sqrt{n}},$$

it follows from (50) that

$$c_n < \sqrt{n}. \tag{51}$$

† The inequality $(1-x^2)^n \ge 1-nx^2$ which we used above is easily shown to be true by considering the function

$$(1-x^2)^n - 1 + nx^2$$

which is zero at $x = 0$ and whose derivative is positive on (0, 1).

For any $\delta > 0$, (51) implies

$$Q_n(x) \le \sqrt{n}(1-\delta^2)^n \qquad (\delta \le |x| \le 1), \tag{52}$$

so that $Q_n \to 0$ uniformly in $\delta \le |x| \le 1$.

Now set

$$P_n(x) = \int_{-1}^{1} f(x+t)Q_n(t)dt \qquad (0 \le |x| \le 1). \tag{53}$$

Our assumptions about f show, by a simple change of variable, that

$$P_n(x) = \int_{-x}^{1-x} f(x+t)Q_n(t)dt$$

$$= \int_{0}^{1} f(t)Q_n(t-x)dt \text{ , by putting } x+t=u$$

and the last integral is clearly a polynomial in x. Thus $\{P_n\}$ is a sequence of polynomials, which are real if f is real.

Given $\varepsilon > 0$, we choose $\delta > 0$ such that $|y-x| < \delta$ implies

$$|f(y) - f(x)| < \frac{\varepsilon}{2}.$$

Let $$M = \sup\{|f(x)| : x \in [a,b]\}.$$

Using (50), (52), and the fact that $Q_n(x) \ge 0$, we see that for $0 \le x \le 1$,

$$|P_n(x) - f(x)| = \left| \int_{-1}^{1} [f(x+t) - f(x)]Q_n(t)dt \right|$$

$$\le \int_{-1}^{1} |f(x+t) - f(x)|Q_n(t)dt$$

$$\le 2M \int_{-1}^{-\delta} Q_n(t)dt + \frac{\varepsilon}{2} \int_{-\delta}^{\delta} Q_n(t)dt + 2M \int_{\delta}^{1} Q_n(t)dt$$

$$\le 4M\sqrt{n}(1-\delta^2)^n + \frac{\varepsilon}{2}$$

$$< \varepsilon$$

for all large enough n, which proves the theorem.

(7.2) Corollary For every interval $[-a, a]$ there is a sequence of real polynomials P_n such that $P_n(0) = 0$ and such that

$$\lim_{n\to\infty} P_n(x) = |x|$$

uniformly on [$-a$, a].

Proof

By Theorem (7.1), there exists a sequence $\left\{P_n^*\right\}$ of real polynomials which converges to $|x|$ uniformly on [$-a$, a]. In particular, $P_n^*(0) \to 0$ as $n \to \infty$. The polynomials

$$P_n(x) = P_n^*(x) - P_n^*(0) \qquad (n = 1, 2, 3, \ldots)$$

have desired properties: $P_n(0) = 0$ and such that

$$\lim_{n\to\infty} P_n(x) = |x|$$

uniformly on [$-a$, a].

(7.3) Definition A family $\mathcal{A}$ of complex functions defined on a set E is said to be an **algebra** (complex algebra) if

(i) $\mathcal{A}$ is closed under addition:

i.e., $f + g \in \mathcal{A} \quad \forall f \in \mathcal{A},\ g \in \mathcal{A}$

(ii) $\mathcal{A}$ is closed under multiplication:

i.e., $fg \in \mathcal{A} \quad \forall f \in \mathcal{A},\ g \in \mathcal{A}$

(iii) $\mathcal{A}$ is closed under scalar multiplication:

i.e., $cf \in \mathcal{A} \quad \forall f \in \mathcal{A}$ and complex constant c.

Example: The set of all polynomials is an algebra.

(7.4) Definition A family $\mathcal{A}$ of complex functions defined on a set E is said to be a **real algebra** if

(i) $\mathcal{A}$ is closed under addition:

i.e., $f + g \in \mathcal{A} \quad \forall f \in \mathcal{A},\ g \in \mathcal{A}$

(ii) $\mathcal{A}$ is closed under multiplicaton:

i.e., $fg \in \mathcal{A} \quad \forall f \in \mathcal{A},\ g \in \mathcal{A}$

(iii) $\mathcal{A}$ is closed under scalar multiplication:

i.e., $cf \in \mathcal{A} \quad \forall f \in \mathcal{A}$ and <u>real</u> constant c.

(7.5) Definition If $\mathcal{A}$ has the property that $f \in \mathcal{A}$ whenever $f_n \in \mathcal{A}$ $(n = 1, 2, 3, \ldots)$ and $f_n \to f$ uniformly on E, then $\mathcal{A}$ is said to be **uniformly closed.**

(7.6) Definition Let $\mathcal{B}$ be the set of all functions which are limits of uniformly convergent sequences of members of $\mathcal{A}$. Then $\mathcal{B}$ is called the **uniform closure** of $\mathcal{A}$.

Remark: Since the set of all polynomials is an algebra, the Weierstrass theorem may be stated by saying that the set of continuous functions on $[a, b]$ is the uniform closure of the set of polynomials on $[a, b]$.

(7.7) Theorem Let $\mathcal{B}$ be the uniform closure of an algebra $\mathcal{A}$ of bounded functions. Then $\mathcal{B}$ is a uniformly closed algebra.

Proof

If $f \in \mathcal{B}$ and $g \in \mathcal{B}$, (by the Definition (7.6) of $\mathcal{B}$) there exist uniformly convergent sequences $\{f_n\}, \{g_n\}$ such that $f_n \to f$, $g_n \to g$ and $f_n \in \mathcal{A}$, $g_n \in \mathcal{A}$. Since we are dealing with bounded functions, it is easy to show that

$$f_n + g_n \to f + g, \quad f_n g_n \to fg, \quad cf_n \to cf,$$

where c is any constant, the convergence being uniform in each case.

Hence (again by the Definition (7.6) of $\mathcal{B}$) $f + g \in \mathcal{B}$, $fg \in \mathcal{B}$, and $cf \in \mathcal{B}$, so that $\mathcal{B}$ is an algebra.

Also $\mathcal{B}$, being the closure of a set, is (uniformly) closed.

(7.8) Definition Let $\mathcal{A}$ be a family of functions on a set E. Then $\mathcal{A}$ is said to **separate points** on E if to every pair of distinct points $x_1, x_2 \in E$ there corresponds a function $f \in \mathcal{A}$ such that $f(x_1) \neq f(x_2)$.

(7.9) Definition Let $\mathcal{A}$ be a family of functions on a set E. If to each $x \in E$ there corresponds a function $g \in \mathcal{A}$ such that $g(x) \neq 0$, we say that $\mathcal{A}$ **vanishes at no point of** E.

(7.10) Examples

- The algebra of all polynomials in one variable is clearly has the properties in Definitions (7.8) and (7.9).

- An example of an algebra which does not separate points is the set of all even polynomials, say on $[-1, 1]$, since $f(-x) = f(x)$ for every even function f, so that for two distinct points x and $-x$ there is no polynomial such that $f(-x) \neq f(x)$.

(7.11) Theorem Suppose $\mathcal{A}$ is an algebra of functions on a set E, $\mathcal{A}$ separates points on E, and $\mathcal{A}$ vanishes at no point of E. Suppose x_1, x_2 are distinct points of E, and c_1, c_2 are constants (real if $\mathcal{A}$ is a real algebra). Then $\mathcal{A}$ contains a function f such that

$$f(x_1) = c_1, \qquad f(x_2) = c_2.$$

Proof

- $\mathcal{A}$ separates points on E, implies that there exist a function $g \in \mathcal{A}$ such that $g(x_1) \neq g(x_2)$.
- $\mathcal{A}$ vanishes at no point of E, hence corresponding to x_1 there exist a function $h \in \mathcal{A}$ such that $h(x_1) \neq 0$.
- $\mathcal{A}$ vanishes at no point of E, hence corresponding to x_2 there exist a function $k \in \mathcal{A}$ such that $k(x_2) \neq 0$.

Define the functions u and v by

$$u = gk - g(x_1)k, \qquad v = gh - g(x_2)h.$$

Then $u \in \mathcal{A}$, $v \in \mathcal{A}$,

$$u(x_1) = g(x_1)k(x_1) - g(x_1)k(x_1) = 0,$$

$$u(x_2) = g(x_2)k(x_2) - g(x_1)k(x_2) = \left[g(x_2) - g(x_1)\right]k(x_2) \neq 0,$$

$$v(x_1) \neq 0, \; v(x_2) = 0.$$

Hence the function

$$f = \frac{c_1 v}{v(x_1)} + \frac{c_2 u}{u(x_2)}$$

has the properties:

$$f(x_1) = \frac{c_1 v(x_1)}{v(x_1)} + \frac{c_2 u(x_1)}{u(x_2)} = c_1 \text{ and } f(x_2) = \frac{c_1 v(x_2)}{v(x_1)} + \frac{c_2 u(x_2)}{u(x_2)} = c_2.$$

(7.12) Theorem ***(Stone's generalization of the Weierstrass theorem)*** Let $\mathcal{A}$ be an algebra of real continuous functions on a compact set K. If $\mathcal{A}$ separates points on K and if $\mathcal{A}$

vanishes at no point of K, then the uniform closure $\mathcal{B}$ of $\mathcal{A}$ consists of all real continuous functions on K.

Proof

The proof is divided into four steps.

Step 1 If $f \in \mathcal{B}$, then $|f| \in \mathcal{B}$.

Proof

Let

$$a = \sup\{|f(x)| : x \in K\} \tag{54}$$

and let $\varepsilon > 0$ be given. By Corollary (7.2) there exist real numbers $c_1, \ldots, c_n$ such that

$$\left|\sum_{i=1}^{n} c_i y^i - |y|\right| < \varepsilon \qquad (-a \le y \le a). \tag{55}$$

Since $\mathcal{B}$ is an algebra, the function

$$g = \sum_{i=1}^{n} c_i f^i$$

is a member of $\mathcal{B}$. By (54) and (55), we have

$$|g(x) - |f(x)|| < \varepsilon \qquad (x \in K).$$

Since $\mathcal{B}$ is uniformly closed, this shows that $|f| \in \mathcal{B}$.

Step 2 If $f \in \mathcal{B}$, and $g \in \mathcal{B}$, then $\max(f, g) \in \mathcal{B}$ and $\min(f, g) \in \mathcal{B}$.

Proof

By $\max(f, g)$ we mean the function h defined by

$$h(x) = \begin{cases} f(x) & \text{if } f(x) \ge g(x) \\ g(x) & \text{if } f(x) < g(x) \end{cases}$$

and is given by

$$h = \frac{f+g}{2} + \frac{|f-g|}{2},$$

and by $\min(f, g)$ we mean the function l defined by

$$l(x) = \begin{cases} f(x) & \text{if } f(x) \le g(x) \\ g(x) & \text{if } f(x) > g(x) \end{cases}$$

and is given by

$$l = \frac{f+g}{2} - \frac{|f-g|}{2}.$$

Step 2 follows from Step 1 and the identities

$$\max(f, g) = \frac{f+g}{2} + \frac{|f-g|}{2},$$

and

$$\min(f, g) = \frac{f+g}{2} - \frac{|f-g|}{2}.$$

By iteration, the result can of course be extended to any finite set of functions: If $f_1, \ldots, f_n \in \mathcal{B}$, then $\max(f_1, \ldots, f_n) \in \mathcal{B}$ and $\min(f_1, \ldots, f_n) \in \mathcal{B}$.

Step 3 Given a real function f, continuous on K, a point $x \in K$, and $\varepsilon > 0$, there exists a function $g_x \in \mathcal{B}$ such that $g_x(x) = f(x)$ and

$$g_x(t) > f(t) - \varepsilon \qquad (t \in K). \tag{56}$$

Proof

Since $\mathcal{A} \subset \mathcal{B}$ and $\mathcal{A}$ satisfies the hypotheses of Theorem (7.11) so does $\mathcal{B}$. Hence, for every $y \in K$, we can find a function $h_y \in \mathcal{B}$ such that

$$h_y(x) = f(x), \qquad h_y(y) = f(y). \tag{57}$$

By the continuity of h, there exist an open set J_y containing y, such that

$$h_y(t) > f(t) - \varepsilon \qquad (t \in J_y). \tag{58}$$

Then the collection $\{J_y : y \in K\}$ form an open cover for K. Since K is compact, this cover has a finite sub cover, and hence there is a finite set of points $y_1, \ldots, y_n$ such that

$$K \subset J_{y_1} \cup \cdots \cup J_{y_n}. \tag{59}$$

Put

$$g_x = \max(h_{y_1}, \ldots, h_{y_n}).$$

By Step 2, $g_x \in \mathcal{B}$, and the relations (57) to (59) show that g_x has the other required properties.

Step 4 Given a real function f, continuous on K, and $\varepsilon > 0$, there exists a function $h \in \mathcal{B}$ such that

$$|h(x) - f(x)| < \varepsilon \qquad (x \in K). \tag{60}$$

Since $\mathcal{B}$ is uniformly closed, this statement is equivalent to the conclusion of the theorem.

Proof

Let us consider the functions g_x, for each $x \in K$, constructed in step 3. By the continuity of g_x, there exist open sets V_x containing x, such that

$$g_x(t) < f(t) + \varepsilon \qquad (t \in V_x). \tag{61}$$

Since K is compact, there exists a finite set of points $x_1, \ldots, x_m$ such that

$$K \subset V_{x_1} \cup \cdots \cup V_{x_m}. \tag{62}$$

Put

$$h = \min\left(g_{x_1}, \ldots, g_{x_m}\right).$$

By step 2, $h \in \mathcal{B}$, and (56) implies

$$h(t) > f(t) - \varepsilon \qquad (t \in K), \tag{63}$$

whereas (61) and (62) imply

$$h(t) < f(t) + \varepsilon \qquad (t \in K). \tag{64}$$

From (63) and (64), we have

$$-\varepsilon < h(t) - f(t) < \varepsilon \qquad (t \in K)$$

i.e.,

$$|h(t) - f(t)| < \varepsilon \qquad (t \in K).$$

Hence (60) is obtained.

(7.13) Definition An algebra $\mathcal{A}$ is said to be **self adjoint** if for every $f \in \mathcal{A}$ its complex conjugate $\bar{f} \in \mathcal{A}$, where $\bar{f}$ is defined by $\bar{f}(x) = \overline{f(x)}$.

(7.14) Theorem Suppose $\mathcal{A}$ is a self adjoint algebra of complex continuous functions on a compact set K, $\mathcal{A}$ separates points on K, and $\mathcal{A}$ vanishes at no point of K. Then the uniform closure $\mathcal{B}$ of $\mathcal{A}$ consists of all complex continuous functions on K. In other words, $\mathcal{A}$ is dense in $\mathscr{C}(K)$.

Proof

Let $\mathcal{A}_{\mathbb{R}}$ be the set of all real functions on K which belong to $\mathcal{A}$.

If $f \in \mathcal{A}$ and $f = u + iv$, with u, v real, then $2u = f + \bar{f}$, and since $\mathcal{A}$ is self adjoint, we see that $u \in \mathcal{A}_{\mathbb{R}}$. If $x_1 \neq x_2$, there exists $f \in \mathcal{A}$ such that $f(x_1) = 1$, $f(x_2) = 0$; hence $0 = u(x_2) \neq u(x_1) = 1$, which shows that $\mathcal{A}_{\mathbb{R}}$ separates points on K. If $x \in K$, then $g(x) \neq 0$ for some $g \in \mathcal{A}$, and there is a complex number λ such that $\lambda g(x) > 0$; if $f = \lambda g$, $f = u + iv$, it follows that $u(x) > 0$; hence $\mathcal{A}_{\mathbb{R}}$ vanishes at no point of K.

Thus $\mathcal{A}_{\mathbb{R}}$ satisfies the hypothesis of Theorem (7.12). It follows that every real continuous function on K lies in the uniform closure of $\mathcal{A}_{\mathbb{R}}$, hence lies in $\mathcal{B}$. If f is a complex continuous function on K, $f = u + iv$, then $u \in \mathcal{B}$, $v \in \mathcal{B}$, hence $f \in \mathcal{B}$. This completes the proof.

Chapter 7

LEBESGUE MEASURE

(1.1) Intervals

The simplest sets of real numbers are the intervals. We define **open interval** (a,b) to be the set $(a,b)=\{x\in\mathbb{R}:a<x<b\}$. We always take $a<b$, but we also consider the infinite intervals $(a,\infty)=\{x\in\mathbb{R}:a<x\}$ and $(-\infty,a)=\{x\in\mathbb{R}:x<b\}$. Some times we write $(-\infty,\infty)$ for $\mathbb{R}$, the set of all real numbers. We define **closed interval** $[a,b]$ to be the set $[a,b]=\{x\in\mathbb{R}:a\le x\le b\}$. For closed intervals we take a and b finite but always assume that $a<b$. The **half open interval** $(a,b]$ to be the set $(a,b]=\{x\in\mathbb{R}:a<x\le b\}$. The **half open interval** $[a,b)$ to be the set $[a,b)=\{x\in\mathbb{R}:a\le x<b\}$.

(1.2) Length of an interval

Let I be an interval of any type above. Then the **length** $l(I)$ of the interval I is defined, as usual, to be the difference of the endpoints of the interval. For example, length of the interval (0, 1) is 1 and that of [0, 1] is also 1.

- Length is an example of a *set function*, that is, a function that associates an extended real number to each set in some collection of sets.
- In the case of length the domain is the collection of all intervals.

We should like to extend the notion of length to more complicated sets than intervals. For instance, we could define the *length* of an open set to be the sum of the lengths the open intervals of which it is composed.

(1.3) Definition The system of real numbers $\mathbb{R}$ can be extended by including two elements $+\infty$ and $-\infty$. This enlarged set is called the **extended real number system.**

(1.4) Definition A set function m that assigns to each set E in some collection $\mathfrak{M}$ of sets of real numbers a nonnegative extended real number $m(E)$, or simply mE, is called the **measure of** E if it satisfies the following properties:

i. mE is defined for each set E of real numbers; that is, $\mathfrak{M}=\mathcal{P}(\mathbb{R})$, the power set of $\mathbb{R}$.

ii. For an interval I, $mI=l(I)$.

iii. If $\langle E_n\rangle$ is a sequence of disjoint sets (for which m is defined), $m\left(\bigcup E_n\right)=\sum mE_n$.

iv. m is translation invariant; that is, if E is a set for which m is defined and if $E+y=\{x+y \mid x\in E\}$, the set obtained by replacing each point x in E by the point $x+y$, then $m(E+y)=mE$.

Unfortunately, it is impossible to construct a set function having all four of these properties. So first property must be weakened so that mE need not be defined for all sets E of real numbers. We shall want mE to be defined for as many sets as possible and will find it convenient to require the family $\mathfrak{M}$ of sets for which m is defined to be a $\sigma-$algebra.

(1.5) Definition We say that m is a **countably additive measure** if

(i) it is a nonnegative extended real valued function whose domain of definition is a $\sigma-$algebra $\mathfrak{M}$ of sets (of real numbers) and

*(ii)*e $m\left(\bigcup E_n\right)=\sum mE_n$ for each sequence $\langle E_n\rangle$ of disjoint sets in $\mathfrak{M}$.

Our goal in the coming sections will be the construction of a countably additive measure which is translation invariant and has the property that $mI=l(I)$ for each interval I.

(1.6) Definition Let A and B two sets. Then $A\setminus B$ is the set of all elements in A that are not in B. i.e., $A\setminus B=\{x: x\in A,\ x\notin B\}$.

In Figure 1, shaded region indicates the set $A\setminus B$.

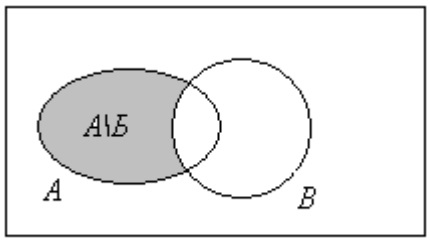


Figure 1

(1.7) Notations

- For a set E, we denote the complement of E by $\tilde{E}$ or some times by $\sim E$..
- $A\setminus B$ is also denoted by $A - B$.

(1.8) Remark $A\setminus B=A\cap\tilde{B}$. [Ref. Figure 1].

(1.9) Definition ***(Algebra)*** A collection $\wp$ of subsets of X is called an **algebra** of sets (or a **Boolean algebra**) if

(i) $A\in\wp,\ B\in\wp \Rightarrow A\cup B\in\wp$.

(ii) $A\in\wp \Rightarrow \tilde{A}\in\wp$.

(1.10) Remark If $\wp$ is an algebra, then

(iii) $A\in\wp,\ B\in\wp \Rightarrow A\cap B\in\wp$.

Proof $A\in\wp,\ B\in\wp \Rightarrow \tilde{A}\in\wp,\ \tilde{B}\in\wp$, by (*ii*) above

$\Rightarrow \tilde{A}\cup\tilde{B}\in\wp$, by (*i*)

$\Rightarrow \left(\tilde{A} \cup \tilde{B}\right) \in \wp$, by (*ii*)

[Here $\left(\tilde{A} \cup \tilde{B}\right)$ denotes the complement of $\tilde{A} \cup \tilde{B}$

$\Rightarrow \tilde{\tilde{A}} \cap \tilde{\tilde{B}} \in \wp$, since $\tilde{\tilde{A}} \cap \tilde{\tilde{B}} = \left(\tilde{A} \cup \tilde{B}\right)$ by De Morgan's law

$\Rightarrow A \cap B \in \wp$, since $\tilde{\tilde{A}} = A$, and $\tilde{\tilde{B}} = B$.

(1.11) Remark If a collection $\wp$ of subsets of X satisfies (*ii*) and (*iii*) above, then by De-Morgan's law (proceeding as in the proof above, replacing $\cap$ by $\cup$ and vice-versa) it also satisfies (*i*) and hence is an algebra.

(1.12) Remark By taking unions two at a time, we see that

(iv) $A_1, \ldots, A_n \in \wp$, then $A_1 \cup \ldots \cup A_n \in \wp$

Also,

(v) $A_1, \ldots, A_n \in \wp$, then $A_1 \cap \ldots \cap A_n \in \wp$.

(1.13) Definition *(σ-Algebra)* An algebra $\wp$ of sets is called a **σ-algebra** (or **Borel field**), if every union of <u>countable collection</u> of sets in $\wp$ is again in $\wp$.

i.e., if $\langle A_i \rangle$ is a sequence of sets, then [in addition to (*i*) to (*ii*) of Definition (1.9)]

(iv) $\bigcup_{i=1}^{\infty} A_i \in \wp$.

By De-Morgan's law, from (*i*) proceeding as in the proof of (*iii*), we have

(v) $\bigcap_{i=1}^{\infty} A_i \in \wp$. (i.e., intersection of countable collection of sets in $\wp$ is again in $\wp$).

(1.14) Theorem Given any collection C of subsets of X, there is a smallest σ-algebra that contains C; i.e., there is a σ-algebra $\wp$ containing C such that if $\mathcal{B}$ is any σ-algebra containing C, then $\wp \subset \mathcal{B}$.

The smallest σ-algebra containing C is called the σ-algebra **generated** by C.

The proof of the theorem is not part of the syllabus and is omitted.

(1.15) Proposition Let $\wp$ be an algebra of subsets and $\langle A_i \rangle$ be a sequence of sets in $\wp$.

Then there is a sequence $\langle B_i \rangle$ of sets in $\wp$ such that $B_n \cap B_m = \Phi$ for $n \neq m$ and

$$\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i.$$

Moreover, $B_1 = A_1$, and for $n > 1$, $B_n = A_n \setminus \left[A_1 \cup A_2 \cup \ldots \cup A_{n-1}\right]$. Hence $B_n \subseteq A_n$ for each *n*.

The proof of the theorem is not part of the syllabus and is omitted.

(1.16) Problems

Let m be a countably additive measure defined for all sets in a $\sigma-$ algebra $\mathfrak{M}$.

1. If A and B are two sets in $\mathfrak{M}$ with $A \subset B$, then $mA \leq mB$. (This property is known as **monotonicity**)

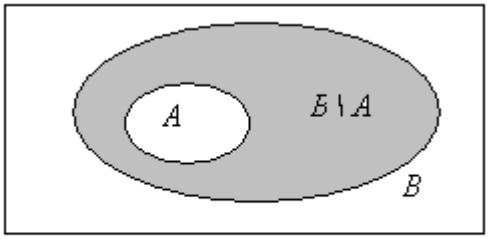


Figure 2

Proof

Since $A \subset B$, we can write

$B = A \cup \left(B \setminus A\right)$. [Ref. Figure 2]

Also note that $A \cup \left(B \setminus A\right) = \Phi$. By

Definition (1.5), since A and $B \setminus A$ are disjoint,

$$m\left[A \cup \left(B \setminus A\right)\right] = mA + m\left(B \setminus A\right)$$

i.e., $mB = mA + m\left(B \setminus A\right)$

implies $mB \geq mA$, since $m\left(B \setminus A\right) \geq 0$ as *m* is nonnegative function.

2. Let $\langle A_i \rangle$ be any sequence of sets in $\mathfrak{M}$. Then $m\left(\bigcup A_i\right) \leq \sum mA_i$.

Proof By Proposition (1.15), there is a sequence $\langle B_i \rangle$ of sets in $\mathfrak{M}$ such that

$B_n \cap B_m = \Phi$ for $n \neq m$ and $\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i$. Hence

$$m\left[\bigcup_{i=1}^{\infty} A_i\right] = m\left[\bigcup_{i=1}^{\infty} B_i\right]$$

$= \sum_{i=1}^{\infty} mB_i$, as *m* is countably additive measure and B_i are mutually disjoint.

$\leq \sum_{i=1}^{\infty} mA_i$, as $B_i \subseteq A_i$ by Proposition (1.15), and by **1** above $B_i \subseteq A_i \Rightarrow mB_i \leq mA_i$.

3. If there is a set A in $\mathfrak{M}$ such that $mA < \infty$, then $m\Phi = 0$.

Proof $A = A \cup \Phi$. Hence

$mA = mA + m\Phi$, as m is countably additive and A and Φ are mutually disjoint.

Since $mA < \infty$, we subtract mA from each side and get $m\Phi = 0$.

4. Let $nE = \begin{cases} \infty, & \text{if } E \text{ is infinite} \\ \text{number of elemets in } E, & \text{if } E \text{ is finite} \end{cases}$

Then n is a countably additive set function that is translation invariant and defined for all sets of real numbers. This measure is called the **counting measure.**

(i) Clearly n is a nonnegative extended real valued function whose domain of definition is a $\sigma-$algebra $\mathfrak{M}$ of sets (of real numbers)

(ii) $n\left(\bigcup E_n\right) = \sum nE_n$ for each sequence $\langle E_n \rangle$ of disjoint sets (E_n's may be finite sets or infinite sets) in $\mathfrak{M}$.

(iii) n is translation invariant; that is, if E is a set for which n is defined and if $E + y = \{x + y \mid x \in E\}$, the set obtained by replacing each point x in E by the point $x + y$, then

Case 1: E is a finite set. Then $E + y$ is also a finite set and contains the same number of element of E. Hence

$nE =$ number of elemets in $E = n(E + y)$.

Case 2: E is an infinite set, then $E + y$ is also infinite. Hence

$$nE = \infty = n(E + y).$$

2 Outer Measure

For each set A of real numbers consider the countable collection $\{I_n\}$ of open intervals that cover A, that is, collection for which $A \subset \bigcup I_n$, and for each such collection consider the sum of the length of the intervals in the collection. Since the lengths are positive numbers, this sum is uniquely defined independently of the order of the terms.

(2.1) Definition Let $\{I_n\}$ of open intervals that cover A. The **outer measure** m^*A of A to be infimum of all sums discussed above. That is,

$$m^*A = \inf_{A \subset \bigcup I_n} \sum l(I_n).$$

(2.2) Remarks

- $m^*\Phi = 0.$
- If $A \subset B,$ then $m^*A \le m^*B.$
- Each set consisting of a single point has outer measure 0.

  Proof Let $\{x\}$ be a singleton set. Then for $\varepsilon > 0$, $(x-\varepsilon,\ x+\varepsilon)$ is an open cover of $\{x\}$ so that $m^*\{x\} = \inf_{A \subset \bigcup I_n} \sum l(I_n) \le 2\varepsilon$. Since ε is arbitrary, it follows that $m^*\{x\} = 0.$

(2.3) Proposition The outer measure of an interval is its length. (UQ)

Proof We begin with the case in which we have a closed finite interval, say $[a,b]$. Since the open interval $(a-\varepsilon,\ b+\varepsilon)$ contains $[a,b]$ for each positive ε, we have

$$m^*[a,b] \le l(a-\varepsilon,\ b+\varepsilon) = b-a+2\varepsilon.$$

Since

$$m^*[a,b] \le b-a+2\varepsilon$$

for each positive ε (in particular for every positive ε near to 0), we must have

$$m^*[a,b] \le b-a.$$

Claim: $m^*[a,b] \ge b-a.$

This is equivalent to show that if $\{I_n\}$ is any countable collection of open intervals covering $[a,b]$, then

$$\sum l(I_n) \ge b-a. \qquad (1)$$

By the Heine-Borel Theorem, any collection of open intervals covering $[a,b]$ contains a finite subcollection that also covers $[a,b]$, and since the sum of the lengths of the finite

subcollection is no greater than the sum of the lengths of the original collection, it suffices to prove the inequality (1) for finite collections $\{I_n\}$ that cover $[a,b]$. Since a is contained in $\bigcup I_n$, there must be one of the I_n's that contains a. Let this be the interval (a_1,b_1). We have $a_1 < a < b_1$. If $b_1 \le b$, then $b_1 \in [a,b]$, and since $b_1 \notin (a_1,b_1)$, there must be an interval (a_2,b_2) in the collection $\{I_n\}$ such that $b_1 \in (a_2,b_2)$; that is $a_2 < b_1 < b_2$. Continuing in this fashion, we obtain a sequence (a_1,b_1), …,(a_k,b_k) from the collection $\{I_n\}$ such that $a_i < b_{i-1} < b_i$.

Since $\{I_n\}$ is a finite collection, our process must terminate with some interval (a_k,b_k). But it terminates only if $b \in (a_k,b_k)$; that is if $a_k < b < b_k$. Thus

$$\begin{aligned}\sum l(I_n) &\ge \sum l(a_i - b_i)\\ &= (b_k - a_k) + (b_{k-1} - a_{k-1}) + \cdots + (b_1 - a_1)\\ &= b_k - (a_k - b_{k-1}) - (a_{k-1} - b_{k-2}) - \cdots - (a_2 - b_1) - a_1\\ &= b_k - a_1,\end{aligned}$$

since $a_i < b_{i-1}$. But $b_k > b$ and $a_1 < a$, and so we have $b_k - a_1 > b - a$, whence $\sum l(I_n) \ge (b-a)$. This shows that $m^*[a,b] = b - a$.

If I is any finite interval, then given $\varepsilon > 0$, there is a closed interval $J \subset I$ such that $l(J) > l(I) - \varepsilon$. Hence

$$l(I) - \varepsilon < l(J) = m^*J \le m^*I \le m^*\bar{I} = l(\bar{I}) = l(I).$$

Thus for each $\varepsilon > 0$,

$$l(I) - \varepsilon < m^*I \le l(I),$$

and so $m^*I = l(I)$.

If I is an infinite interval, then given any real number Δ, there is a closed interval $J \subset I$ with $l(J) = \Delta$. Hence $m^*I \ge m^*J = l(J) = \Delta$. Since $m^*I \ge \Delta$ for each Δ, $m^*I = \infty = l(I)$.

(2.4) Proposition Let $\{A_n\}$ be a countable collection of sets of real numbers. Then

$$m^*\left(\bigcup A_n\right) \le \sum m^*A_n. \tag{1}$$

(UQ 2008)

Property (1) is called *subadditivity.*

Proof

If one of the sets A_n has infinite outer measure, say $m^*A_n = \infty$, the inequality holds trivially.

If m^*A_n is finite for all n, then, given $\varepsilon > 0$, there is a countable collection $\{I_{n,i}\}_i$ of open intervals such that $A_n \subset \bigcup_i I_{n,i}$ and $\sum_i l(I_{n,i}) \le m^*A_n + 2^{-n}\varepsilon$.

Now the collection $\{I_{n,i}\}_{n,i} = \bigcup_n \{I_{n,i}\}_i$ is countable, being the union of a countable number of countable collections, and covers $\bigcup A_n$. Thus

$$m^*\left(\bigcup A_n\right) \le \sum_{n,i} l(I_{n,i}) = \sum_n \sum_i l(I_{n,i}) \le \sum_n \left(m^*A_n + \varepsilon 2^{-n}\right)$$

$$= \sum m^*A_n + \varepsilon.$$

Since ε was an arbitrary positive number,

$$m^*\left(\bigcup A_n\right) \le \sum_n m^*A_n.$$

(2.5) Corollary If A is countable, $m^*A = 0$. (UQ 2008)

Proof

If A is countable, then the set A is of the form $A = \{x_1, x_2, x_3, \ldots, x_n, \ldots\}$. Then A can be expressed as the countable union of pariwise disjoint singleton sets $\{x_1\}, \{x_2\}, \{x_3\}, \ldots, \{x_n\}, \ldots$.

i.e., $A = \{x_1, x_2, x_3, \ldots, x_n, \ldots\} = \{x_1\} \cup \{x_2\} \cup \{x_3\} \cup \ldots \cup \{x_n\} \cup \ldots$

Now by Proposition (2.4),

$$m^*A = m^*\left(\bigcup_n \{x_n\}\right) \le \sum_n m^*\{x_n\} = \sum_n 0 = 0, \text{ since } m^*\{x_n\} = 0 \text{ for } n = 1, 2, 3, \ldots$$

(2.6) Corollary The set $[0, 1]$ is not countable.

Proof

If $[0, 1]$ is countable, then by Corollary (2.5), its measure must be 0 which is false. Hence $[0, 1]$ is not countable.

(2.7) Proposition Given any set A and any $\varepsilon > 0$, there is an open set O such that $A \subset O$ and $m^*O \le m^*A + \varepsilon$. There is a $G \in G_\delta$ such that $A \subset G$ and $m^*A = m^*G$.

(2.8) Problems

1. Let A be the set of rational numbers between 0 and 1, and let $\{I_n\}$ be a finite collection of open intervals covering A. Then $\sum l(I_n) \ge 1$.

Here $A = (0, 1) \cap \square$. Let $\{I_n\}$ be a finite collection of open intervals covering A. Since A is dense in $(0, 1)$, $\{I_n\}$ will be a cover for $(0, 1)$ also. Now

$$m^*(0, 1) = \inf_{A \subset \bigcup J_n} \sum l(J_n),$$

infimum taken over all open covering $\{J_n\}$ of $(0, 1)$. Since $\{I_n\}$ is a cover for $(0, 1)$, the above infimum will not be greater than $\sum l(I_n)$.

If $\sum l(I_n) < 1$, the above forces us to have $m^*(0, 1) < 1$, which is a contradiction to the fact that $m^*(0, 1) = 1$, by Proposition (2.3). Hence $\sum l(I_n) \ge 1$.

2. Prove that m^* is translation invariant.

Proof Note that if $\{x + I_n\}$ is a cover of $x + A$, then $\{I_n\}$ is a cover of A and $l(x + I_n) = l(I_n)$. Hence

$$m^*(x + A) = \inf_{x + A \subset x + \bigcup I_n} \sum l(x + I_n) = \inf_{A \subset \bigcup I_n} \sum l(I_n) = m^*(A).$$

3. Prove that if $m^*A = 0$, then $m^*(A \cup B) = m^*B$.

Proof Suppose $m^*A = 0$. Then, since $B \subset A \cup B$, we have

$$m^*B \le m^*(A \cup B) \le m^*A + m^*B = m^*B \qquad \text{implies} \qquad m^*(A \cup B) = m^*B.$$

3 Measurable Sets

(2.9) Definition A set E is said to be **measurable** if for each set A we have

$$m^*A = m^*(A \cap E) + m^*(A \cap \tilde{E}).$$

(2.10) Remarks

- Since we always have $m^*A \le m^*(A\cap E)+m^*(A\cap \tilde{E})$, we see that E is measurable if (and only if) for each A we have $m^*A \ge m^*(A\cap E)+m^*(A\cap \tilde{E})$.
- Since the definition of measurability is symmetric in E and $\tilde{E}$, we have $\tilde{E}$ measurable whenever E is.
- Clearly Φ and $\square$ are measurable.

(2.11) Lemma If $m^*E = 0$, then E is measurable.

Proof

Suppose $m^*E = 0$. Let A be any set. Then $A\cap E \subset E$, and so $m^*(A\cap E) \le m^*E = 0$. Also $A \supset A\cap \tilde{E}$, and so

$$m^*A \ge m^*(A\cap \tilde{E}) = m^*(A\cap \tilde{E})+m^*(A\cap E),$$

and therefore, by the Remark above, E is measurable.

(2.12) Lemma If E_1 and E_2 are measurable, so is $E_1 \cup E_2$.

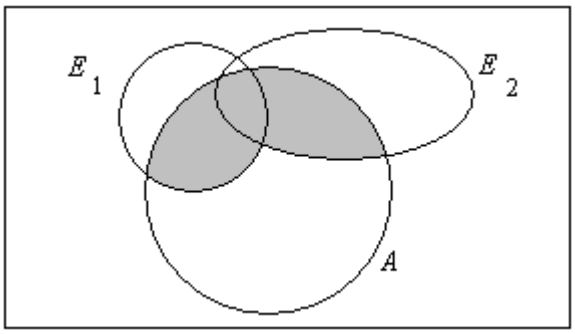
Proof

Let A be any set. Since E_2 is measurable, we have

$$m^*(A\cap \tilde{E}_1) = m^*(A\cap \tilde{E}_1\cap E_2)+m^*(A\cap \tilde{E}_1\cap \tilde{E}_2),$$

and since $A\cap(E_1\cup E_2) = [A\cap E_1]\cup[A\cap E_2\cap \tilde{E}_1]$, [Ref. Figure] we have

$$m^*(A\cap(E_1\cup E_2)) \le m^*(A\cap E_1)+m^*(A\cap E_2\cap \tilde{E}_1).$$



Thus

$$m^*(A\cap(E_1\cup E_2))+m^*(A\cap \tilde{E}_1\cap \tilde{E}_2) \le m^*(A\cap E_1)+m^*(A\cap E_2\cap \tilde{E}_1)+m^*(A\cap \tilde{E}_1\cap \tilde{E}_2)$$
$$\le m^*(A\cap E_1)+m^*(A\cap \tilde{E}_1) = m^*A,$$

by the measurability of E_1. Since $\sim(E_1 \cup E_2) = \tilde{E}_1 \cap \tilde{E}_2$, this shows that $E_1 \cup E_2$ is measurable.

(2.13) Corollary to Lemma (2.12) The family $\mathcal{M}$ of measurable sets is an algebra of sets.

Proof

(i) $E_1 \in \mathcal{M}$ and $E_2 \in \mathcal{M}$ implies E_1 and E_2 are measurable and by Lemma (2.12) $E_1 \cup E_2$ is measurable, i.e., $E_1 \cup E_2 \in \mathcal{M}$.

*(ii)*By the second remark in (2.10), we have E is measurable implies $\tilde{E}$ is measurable. i.e., $E \in \mathcal{M}$ implies $\tilde{E} \in \mathcal{M}$.

Hence, by Definition (1.9), $\mathcal{M}$ is an algebra.

(2.14) Corollary to (2.15) If E_1 and E_2 measurable, then $E_1 \sim E_2$ is measurable.

Proof E_1 and E_2 measurable implies $E_1 \in \mathcal{M}$ and $E_2 \in \mathcal{M}$ implies (since $\mathcal{M}$ an algebra of sets) $E_1 \in \mathcal{M}$ and $\tilde{E}_2 \in \mathcal{M}$ implies $E_1 \cap \tilde{E}_2 \in \mathcal{M}$ [Ref Remark (1.10)] implies $E_1 \sim E_2 \in \mathcal{M}$, since $E_1 \sim E_2 = E_1 \cap \tilde{E}_2$.

(2.16) Lemma Let A be any set, and $E_1, \ldots, E_n$ a finite sequence of disjoint measurable sets. Then

$$m^*\left(A \cap \left[\bigcup_{i=1}^{n} E_i\right]\right) = \sum_{i=1}^{n} m^*(A \cap E_i).$$

Proof We prove the Lemma by induction on n. It is clearly true for $n = 1$, and we assume it is true if we have $n - 1$ sets E_i. Since the E_i are disjoint sets, we have

$$A \cap \left[\bigcup_{i=1}^{n} E_i\right] \cap E_n = A \cap E_n$$

and

$$A \cap \left[\bigcup_{i=1}^{n} E_i\right] \cap \tilde{E}_n = A \cap \left[\bigcup_{i=1}^{n-1} E_i\right].$$

Hence the measurability of E_n implies that

$$m^*\left(A \cap \left[\bigcup_{i=1}^{n} E_i\right]\right) = m^*(A \cap E_n) + m^*\left(A \cap \left[\bigcup_{i=1}^{n-1} E_i\right]\right)$$

$$= m^*(A \cap E_n) + \sum_{i=1}^{n-1} m^*(A \cap E_i),$$

by our assumption of Lemma for $n - 1$ sets.

(2.17) Theorem The collection $\mathcal{M}$ of measurable sets is a σ-algebra; that is, the complement of a measurable set is a measurable and the union (and intersection) of a countable collection of measurable sets is measurable. Moreover, every set with outer measure zero is measurable.

Proof We have already observed in Corollary (2.13) that $\mathcal{M}$ is an algebra of sets, and so we have only to prove that if a set E is the union of a countable collection of measurable sets it is measurable. By Proposition such an E must be union of a sequence $\{E_n\}$ of pairwise disjoint measurable sets. Let A be any set, and let $F_n = \bigcup_{i=1}^{n} E_i$. Then F_n is measurable, and $\tilde{F}_n \supset \tilde{E}$. Hence

$$m^*A = m^*\left(A \cap F_n\right) + m^*\left(A \cap \tilde{F}_n\right) \geq m^*\left(A \cap F_n\right) + m^*\left(A \cap \tilde{E}\right).$$

By Lemma (2.16)

$$m^*\left(A \cap F_n\right) = \sum_{i=1}^{n} m^*(A \cap E_i).$$

Thus

$$m^*A \geq \sum_{i=1}^{n} m^*(A \cap E_i) + m^*(A \cap \tilde{E}).$$

Since the left side of this inequality is independent of n, we have

$$m^*A \geq \sum_{i=1}^{\infty} m^*(A \cap E_i) + m^*(A \cap \tilde{E})$$

$$\geq m^*(A \cap E) + m^*(A \cap \tilde{E})$$

by the countable subadditivitiy of m^*.

(2.18) Lemma The interval (a, ∞) is measurable.

Proof

Take $E = (a, \infty)$, then $\tilde{E} = (-\infty, a]$. For any set A, we have to show that

$$m^*A = m^*\left(A \cap E\right) + m^*\left(A \cap \tilde{E}\right).$$

i.e., to show that $m^*A = m^*\left(A \cap (a, \infty)\right) + m^*\left(A \cap (-\infty, a]\right).$

Let A be any set, $A_1 = A \cap (a, \infty)$, $A_2 = A \cap (-\infty, a]$.

Clearly, $m^*A \leq m^*A_1 + m^*A_2.$

It remains to show that

$$m^* A_1 + m^* A_2 \le m^* A.$$

If $m^* A = \infty,$ then there is nothing to prove.

If $m^* A < \infty,$ then, given $\varepsilon > 0,$ there is a countable collection $\{I_n\}$ of open intervals which cover A and for which

$$\sum l(I_n) = m^* E + \varepsilon.$$

Let $I_n' = I_n \cap (a, \infty),\ I_n'' = I_n \cap (-\infty, a].$ Then I_n' and I_n'' are intervals (or empty) and

$$l(I_n) = l(I_n') + l(I_n'') = m^* I_n' + m^* I_n''.$$

Since $A_1 \subset \bigcup I_n',$ we have

$$m^* A_1 \le m^* \left(\bigcup I_n' \right) \le m^* I_n',$$

and, since $A_2 \subset \bigcup I_n'',$ we have

$$m^* A_2 \le m^* \left(\bigcup I_n'' \right) \le m^* I_n''.$$

Thus

$$m^* A_1 + m^* A_2 \le \sum \left(m^* I_n' + m^* I_n'' \right)$$

$$\le \sum l(I_n) \le m^* A + \varepsilon.$$

But ε was an arbitrary positive number, and so we must have $m^* A_1 + m^* A_2 \le m^* A.$

This completes the proof.

BOREL SET

The union of a any collection of open sets is open, the intersection of any *finite* collection of open sets is open, but the intersection of a *countable* collection of open sets need not be open. For example: For each $n \in \square$, the set $\left(a - \frac{1}{n},\ a + \frac{1}{n} \right)$ is open, but $\bigcap_{n=1}^{\infty} \left(a - \frac{1}{n},\ a + \frac{1}{n} \right) = \{a\}$ is not open.

Similarly, the intersection of any collection of closed sets is closed and the union of any *finite* collection of closed sets is closed, but the union of a *countable* collection of closed sets need not be closed. For example:

The set of rational numbers is the union of a countable collection of closed sets each of which contains exactly one rational number. i.e.,

$$\mathbb{Q} = \bigcup_{a \in \mathbb{Q}} \{a\}.$$

Though each $\{a\}$ is closed, $\mathbb{Q}$ is not closed (For, we can find a sequence in $\mathbb{Q}$ which converges to $\sqrt{2} \notin \mathbb{Q}$.)

Thus if we are interested in σ-algebras of sets that contain all of the closed sets, we must consider more general types of sets than the open and closed sets. This leads us to the following definition:

(2.19) Definition (*Borel set)* The collection $\mathcal{B}$ of **Borel sets** is the smallest σ-algebra which contains all of the open sets.

Such a smallest σ-algebra exists by Proposition (1.14).

(2.20) Remark

- The collection $\mathcal{B}$ of Borel sets is the smallest σ-algebra that contains all closed sets and the smallest σ-algebra that contains the open intervals.

(2.21) Definition A set which is a countable union of closed sets is called an F_σ (F for closed, σ for sum).

(2.22) Examples/Remarks

- Every countable set is an F_σ-set. In particular, every closed set is an F_σ-set.
- A countable union of sets in F_σ is again in F_σ.
- Each open interval is in F_σ, because any open interval (a, b) can be written as

$$(a, b) = \bigcup_{n=1}^{\infty} \left[a + \frac{1}{n}, b - \frac{1}{n} \right].$$

Hence, also, each open set is in F_σ.

(2.23) Definition A set is a G_δ if it is the countable intersection of open sets (G for open, δ for *durchschnitt*).

(2.24) Examples/Remarks

- The complement of an F_σ is a G_δ, and conversely.
- The F_σ and G_δ are Borel sets.

(2.25) Theorem Every Borel set is measurable. In particular,

- each open set is measurable.
- each closed set is measurable.

(UQ 2006)

Proof

(a,∞) is measurable by Lemma (2.18). Since $\mathcal{M}$ is a σ-algebra, the complement of (a,∞) is also measurable. i.e., $(-\infty,a] = \sim(a,\infty)$ is measurable. In particular, for each $n \in \mathbb{N}$ the intervals $\left(-\infty,\ b-\frac{1}{n}\right]$ are measurable and hence elements in $\mathcal{M}$. Since $\mathcal{M}$ is a σ-algebra, their countable union $\bigcup_{n=1}^{\infty}\left(-\infty,\ b-\frac{1}{n}\right]$ is again in $\mathcal{M}$. Since $(-\infty,\ b)$ can be written as

$$(-\infty,\ b) = \bigcup_{n=1}^{\infty}\left(-\infty,\ b-\frac{1}{n}\right],$$

we can say that $(-\infty,\ b)$ is in $\mathcal{M}$. i.e., $(-\infty,\ b)$ measurable.

- An <u>open interval</u> $(a,\ b)$ can be written as $(a,\ b) = (-\infty,\ b) \cap (a,\ \infty)$. By the discussion above $(-\infty,\ b)$ and $(a,\ \infty)$ are in $\mathcal{M}$ and since $\mathcal{M}$ is an algebra their intersection is again in $\mathcal{M}$. Hence $(a,\ b)$ is in $\mathcal{M}$. i.e., each open interval $(a,\ b)$ is measurable.
- Each <u>open set</u> is the union of a countable number of open intervals. By the discussion above each open interval is in $\mathcal{M}$ and since $\mathcal{M}$ is a σ-algebra, the countable union of open intervals is in $\mathcal{M}$. Hence each open set is measurable.
- Each <u>closed set</u> is the complement of an open set. By the discussion above each open set is in $\mathcal{M}$ and since $\mathcal{M}$ is a σ-algebra, the complement of open set must be in $\mathcal{M}$. Hence each closed set is measurable.

Thus $\mathcal{M}$ is a σ-algebra containing the open sets and must therefore contain the family $\mathcal{B}$ of Borel sets, since $\mathcal{B}$ is the smallest σ-algebra containing the open sets [Ref. Definition (2.19)]

Note: The theorem also follows immediately from the fact that $\mathcal{M}$ is a σ-algebra containing each interval of the form $(a,\ \infty)$ and the fact that $\mathcal{B}$ is the smallest σ-algebra containing all such intervals.

4 Properties of Lebesgue Measure

(3.1) Definition If E is a measurable set, we define the **Lebesgue measure** mE to be the outer measure of E. Thus m is the set function obtained by restricting the set function m^* to the family $\mathcal{M}$ of measurable sets.

(3.2) Remark

- $mE = \inf_{E \subset \bigcup I_n} \sum l(I_n)$.

(3.3) Proposition Let $\langle E_i \rangle$ be a sequence of measurable sets. Then

$$m\left(\bigcup E_i\right) \leq \sum mE_i. \tag{1}$$

If the sets E_n are pairwise disjoint, then

$$m\left(\bigcup E_i\right) = \sum mE_i. \tag{2}$$

Property (1) is called the countable suadditivity.

Proof The inequality is simply a restatement of the subadditivity of m^* given
By Proposition (2.4), we have

$$m^*\left(\bigcup A_n\right) \leq \sum m^* A_n \tag{3}$$

for a countable collection $\{A_n\}$ of sets of real numbers. Here $\langle E_i \rangle$ be a sequence of measurable sets and by Definition (3.1) m is the restriction of the set function m^* to the family $\mathcal{M}$ of measurable sets; hence (3) becomes

$$m\left(\bigcup E_i\right) \leq \sum mE_i.$$

It remains to show that if the sets E_n are pairwise disjoint, then

$$m\left(\bigcup E_i\right) = \sum mE_i.$$

Case 1) If $\langle E_i \rangle$ is a finite sequence of disjoint measurable sets, then Lemma (2.16) with $A = \square$ implies that

$$m\left(\bigcup E_i\right) = \sum mE_i,$$

and so $\mathcal{M}$ is finitely additive.

Case 2) Let $\langle E_i \rangle$ be an infinite sequence of pairwise disjoint measurable sets. Then

$$\bigcup_{i=1}^{\infty} E_i \supset \bigcup_{i=1}^{n} E_i,$$

and so

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) \geq m\left(\bigcup_{i=1}^{n} E_i\right)$$

$$= \sum_{i=1}^{n} mE_i, \text{ by Case 1 above.}$$

Since the left side of this inequality is independent of n, we have

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) \geq \sum_{i=1}^{\infty} mE_i.$$

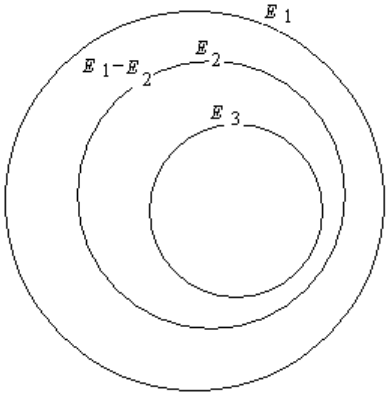
i.e.,
$$\sum_{i=1}^{\infty} mE_i \leq m\left(\bigcup_{i=1}^{\infty} E_i\right). \tag{4}$$

The reverse inequality follows from countable subadditivity (1), and hence from (1) and (4) we have

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{n} mE_i.$$

(3.4) Proposition Let $\langle E_n \rangle$ be an infinite decreasing sequence of measurable sets, that is, a sequence with $E_{n+1} \subset E_n$ for each n. Let mE_1 be finite. Then

$$m\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} mE_n.$$



Proof Let $E = \bigcap_{i=1}^{\infty} E_i$, and let $F_i = E_i \setminus E_{i+1}$. Then

$$E_1 \sim E = \bigcup_{i=1}^{\infty} F_i,$$

and the sets F_i are pairwise disjoint. Hence

$$m(E_1 \sim E) = m\left(\bigcup_{i=1}^{\infty} F_i\right)$$

$$= \sum_{i=1}^{\infty} mF_i, \text{ as } F_i \text{ are pairwise disjoint and by (2) of Prop(3.3)}$$

$$= \sum_{i=1}^{\infty} m(E_i \sim E_{i+1}).$$

Since $E = \bigcap_{i=1}^{\infty} E_i$, $E \subset E_1$ and hence we can write $E_1 = E \cup (E_1 \sim E)$ where E and $(E_1 \sim E)$ are pairwise disjoint. Hence

$$mE_1 = mE + m(E_1 \sim E).$$

Similarly, since $E_{i+1} \subset E_i$, we can write $E_i = E_{i+1} \cup (E_i \sim E_{i+1})$ where E_{i+1} and $E_i \sim E_{i+1}$ are pairwise disjoint and hence

$$mE_i = mE_{i+1} + m(E_i \sim E_{i+1}).$$

Since $mE_i \le mE_1 < \infty$, we have

$$m(E_1 \sim E) = mE_1 - mE$$

and

$$m(E_i \sim E_{i+1}) = mE_i - mE_{i+1}.$$

Thus

$$\begin{aligned} mE_1 - mE &= \sum_{i=1}^{\infty} (mE_i - mE_{i+1}) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} (mE_i - mE_{i+1}) \\ &= \lim_{n\to\infty} (mE_1 - mE_n) \\ &= mE_1 - \lim_{n\to\infty} mE_n. \end{aligned}$$

Since $mE_1 < \infty$, we have

$$mE = \lim_{n\to\infty} mE_n.$$

(3.5) Definition Let A, B two sets. Then the **symmetric difference** of A and B is the set given by

$$A \Delta B = (A \sim B) \cup (B \sim A).$$

(3.6) Proposition (*First Principle of Littlewood*) Let E be a given set. Then the following five statements are equivalent:

i. E is measurable.

ii. Given $\varepsilon > 0$, there is an open set $O \supset E$ with $m^*(O \sim E) < \varepsilon$.

iii. Given $\varepsilon > 0$, there is a closed set $F \subset E$ with $m^*(E \sim F) < \varepsilon$.

iv. There is a G in G_δ with $E \subset G$, $m^*(G \sim E) = 0$.

v. There is a F in F_σ with $F \subset E$, $m^*(E \sim F) = 0$.

If m^*E is finite, the above statements are equivalent to:

vi. Given $\varepsilon > 0$, there is a finite union U of open intervals such that $m^*(U \Delta E) < \varepsilon$.

Proof

(*i*) implies (*ii*): Let E be a measurable set of finite measure. Since $m^*A = \inf_{A \subset \bigcup I_n} \sum l(I_n)$, for $\varepsilon > 0$, $m^*E + \varepsilon$ cannot be a lower bound of the set $\left\{\sum l(I_n) : A \subset \bigcup I_n\right\}$ and hence there exists a countable collection $\{I_n\}$ of open intervals such that $\sum l(I_n) < m^*E + \varepsilon$.

Let $O = \bigcup_{n=1}^{\infty} I_n$. Then $m^*O \le \sum l(I_n) < m^*E + \varepsilon$.

Since $O = E \cup (O \setminus E)$ a disjoint union, we have $m^*O = m^*E + m^*(O \setminus E)$. Thus $m^*(O \setminus E) = m^*O - m^*E < \varepsilon$.

Now we assume that E is a measurable set of infinite measure. For each $n \in \mathbb{N}$, let $E_n = E \cap (-n, n)$. Then each E_n has finite measure and by what proved above there is an open set, say O_n, $O_n \supset E_n$ such that $m^*(O_n \setminus E_n) < \frac{\varepsilon}{2^n}$. Let $O = \bigcup_{n=1}^{\infty} O_n$. Then O is open, $O \supset E$ and since $O \sim E \subset \bigcup_{n=1}^{\infty} O_n \sim E_n$ we get $m^*(O \sim E) \subset m^*\left(\bigcup_{n=1}^{\infty} O_n \sim E_n\right) = \sum_{n=1}^{\infty} m^*(O_n \sim E_n) < \frac{\varepsilon}{2^n} < \varepsilon$. Hence (*i*) implies (*ii*) is proved.

(*ii*) implies (*iii*): E is measurable. Hence its complement $\tilde{E}$ is measurable. Then by (*ii*), there is an open set O such that $O \supset \tilde{E}$ with $m^*(O \sim \tilde{E}) < \varepsilon$. Take $F = \tilde{E}$. Then $F \subset E$ and $O \sim \tilde{E} = E \sim F$. Hence $m^*(E \sim F) < \varepsilon$. i.e., (*ii*) implies (*iii*) is proved.

Similarly, taking complements, (*iii*) implies (*ii*) is proved.

(*ii*) implies (*iv*): By (*ii*) for each $n \in \mathbb{N}$ we can find an open set $O_n \supset E$ such that $m^*(O_n \sim E) < \frac{1}{n}$. Let $G = \bigcap_{n=1}^{\infty} O_n$. Then G is a G_δ set, $E \subset G$ and

$$m^*(G \sim E) \le m^*(O_n \sim E) < \frac{1}{n}.$$

This is true for each n. So $m^*(G \sim E) = 0$.

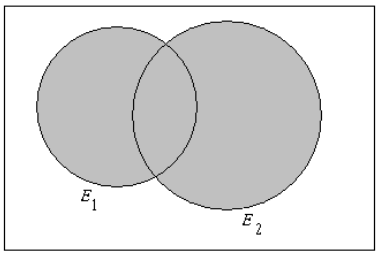
(*iii*) implies (*v*) and (*v*) implies (*iii*) similarly.

(*i*) implies (*vi*): Suppose E is measurable. Then by (*ii*), given $\varepsilon > 0,$ there is an open set $O \supset E$ with $m^*(O \sim E) < \frac{\varepsilon}{2}.$ The open set O can be considered as the disjoint union of open intervals $I_i,\ i = 1, 2, \ldots$ i.e., $O = \bigcup_{i=1}^{\infty} I_i.$ $m^*O = m^*\left(\bigcup_{i=1}^{\infty} I_i\right) = \sum_{i=1}^{\infty} m^* I_i = \sum_{i=1}^{\infty} l(I_i).$ Since m^*O is finite, given $\varepsilon > 0,$ there exists n such that $\sum_{i=n+1}^{\infty} l(I_i) < \frac{\varepsilon}{2}.$ We write $U = \bigcup_{i=1}^{\infty} I_i.$

Then
$$E \Delta U = (U \sim E) \cup (E \sim U)$$
$$= \left(\bigcup_{i=1}^{\infty} I_i \sim E\right) \cup \left(E \sim \bigcup_{i=1}^{\infty} I_i\right)$$
$$\subseteq (O \sim E) \cup \left(O \sim \bigcup_{i=1}^{\infty} I_i\right).$$

Hence
$$m^*(E \Delta U) \le m^*(O \sim E) + m^*\left(O \sim \bigcup_{i=1}^{\infty} I_i\right)$$
$$\le m^*(O \sim E) + m^*\left(\bigcup_{i=1}^{n+1} I_i\right)$$
$$\le m^*(O \sim E) + \sum_{i=1}^{\infty} l(I_i) < \varepsilon.$$

(3.7) Problem Show that if E_1 and E_2 are measurable, then



$m(E_1 \cup E_2) + m(E_1 \cap E_2) = mE_1 + mE_2.$

Solution

$E_1 \cup E_2 = (E_1 \sim E_2) \cup (E_1 \cap E_2) \cup (E_2 \sim E_1)$ is a pairwise disjoint union.

$E_1 = (E_1 \sim E_2) \cup (E_1 \cap E_2)$ is a pairwise disjoint union.

$E_2 = (E_1 \cap E_2) \cup (E_2 \sim E_1)$ is a pairwise disjoint union.

Hence

$$m(E_1 \cup E_2) = m(E_1 \sim E_2) + m(E_1 \cap E_2) + m(E_2 \sim E_1) \quad (1)$$

$$mE_1 = m(E_1 \sim E_2) + m(E_1 \cap E_2) \quad (2)$$

$$mE_2 = m(E_1 \cap E_2) + m(E_2 \sim E_1) \quad (3)$$

Adding (2) and (3), we have

$$mE_1 + mE_2 = m(E_1 \sim E_2) + m(E_1 \cap E_2) + m(E_1 \cap E_2) + m(E_2 \sim E_1)$$

implies

$$mE_1 + mE_2 = m(E_1 \cup E_2) + m(E_1 \cap E_2), \text{ using (1).}$$

Next example says that there is an uncountable set having measure 0.

(3.8) Example The cantor ternary set has measure zero. (UQ 2006)

Proof

We first describe Cantor set.

Let $E_0 = [0, 1]$. Remove the segment $\left(\frac{1}{3}, \frac{2}{3}\right)$, and let $E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Remove the middle thirds of these intervals, and let

$$E_2 = [0, \tfrac{1}{9}] \cup [\tfrac{2}{9}, \tfrac{3}{9}] \cup [\tfrac{6}{9}, \tfrac{7}{9}] \cup [\tfrac{8}{9}, 1].$$

Continuing in this way, we obtain a sequence of compact sets E_n, such that

(a) $E_1 \supset E_2 \supset E_3 \supset \cdots$;

(b) E_n is the union of 2^n intervals, each of length 3^{-n}.

The set

$$P = \bigcap_{n=1}^{\infty} E_n$$

is called the **Cantor set.** P is clearly compact, and by the Theorem "If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, then $\bigcap_\alpha K_\alpha$ is nonempty" P is nonempty.

Now we come to our problem:

Clearly $$m^* E_n = 2^n \left(\frac{1}{3}\right)^n = \left(\frac{2}{3}\right)^n \qquad (n = 1, 2, 3, \ldots);$$

Since $$P = \bigcap_{n=1}^{\infty} E_n,$$ we have

$$P \subset E_n \qquad \forall\ n,$$

so that $$m^* P \le \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0.$$

i.e., the measure of cantor ternary set is zero.

5 Existence of Nonmeasurable set

In this section we are going to show the existence of a nonmeasurable set.

(4.1) Definition If x and y are real numbers in [0, 1), we define the **sum modulo 1** of x and y, denoted by $x \overset{\circ}{+} y$, as follows:

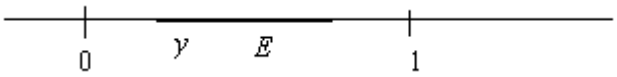
$$x \overset{\circ}{+} y = \begin{cases} x+y, & \text{if } x+y<1 \\ x+y-1, & \text{if } x+y \geq 1 \end{cases}$$

- $\overset{\circ}{+}$ is a commutative and associative binary operation taking pairs of numbers in [0, 1) into numbers in [0, 1).

(4.2) Lemma Let $E \subset [0, 1)$ be a measurable set. Then for each $y \in [0, 1)$ the set $E \overset{\circ}{+} y$ is measurable and $m\left(E \overset{\circ}{+} y\right) = mE.$

Proof

Take $y \in [0, 1)$. Let $E_1 = E \cap [0, 1-y)$ and $E_2 = E \cap [1-y, 1).$



Then E_1 and E_2 are disjoint measurable sets whose union is E, and so

$$mE = mE_1 + mE_2.$$

Now $E_1 \overset{\circ}{+} y = E_1 + y$, and so $m\left(E_1 \overset{\circ}{+} y\right) = m\left(E_1 + y\right) = mE_1,$ since m is translation invariant, and so $E_1 \overset{\circ}{+} y$ is measurable. Also $E_2 \overset{\circ}{+} y = E_1 + (y-1)$ and so $m\left(E_2 \overset{\circ}{+} y\right) = m\left(E_2 + (y-1)\right) = mE_2,$ and so $E_2 \overset{\circ}{+} y$ is also measurable.

But $E \overset{\circ}{+} y = \left(E_1 \overset{\circ}{+} y\right) \cup \left(E_2 \overset{\circ}{+} y\right)$ and the sets $E_1 \overset{\circ}{+} y$ and $E_2 \overset{\circ}{+} y$ are disjoint measurable sets, so by Lemma (2.12), $E \overset{\circ}{+} y$ is measurable. Also,

$$\begin{aligned} m\left(E \overset{\circ}{+} y\right) &= m\left(E_1 \overset{\circ}{+} y\right) + m\left(E_2 \overset{\circ}{+} y\right), \text{ since } E_1 \overset{\circ}{+} y \text{ and } E_2 \overset{\circ}{+} y \text{ are disjoint} \\ &= mE_1 + mE_2 \\ &= mE. \end{aligned}$$

This completes the proof.

(4.3) Construction of a nonmeasuarble set

Step 1: We define a relation on [0, 1) as follows:

$x \sim y$ if $x - y$ is a rational number. It is easy to see that ~ is an equivalence relation. Hence ~ partitions [0, 1) into equivalence classes, i.e., classes such that any two elements of one class differ by a rational number, while any two elements of different classes differ by an irrational number (Ref. Figure). By the axiom of choice[1] there is a set P which contains exactly one element from each equivalence class.

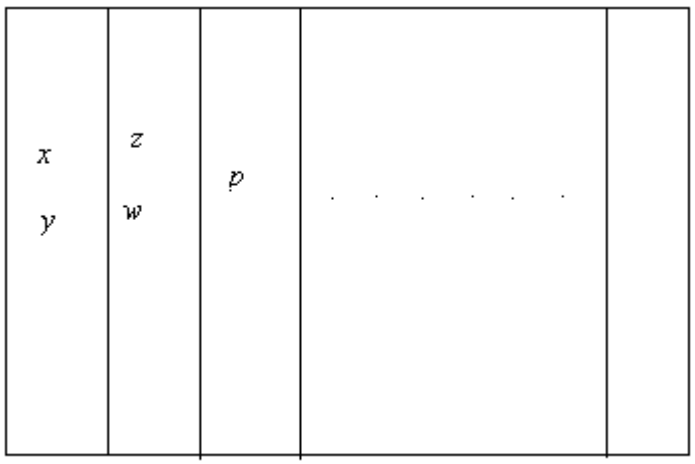


Figure: Partition of [0, 1) by the equivalence relation ~

In the above figure, $x \sim y$ and (x and y are in the same classe) $x - y$ is a rational number; $x \not\sim z$ and (x and z are in different classes) $x - z$ is an irrational number; $z \sim w$ and $z - w$ is a rational number; $z \not\sim p$ and $z - p$ is an irrational number.

Step 2: Let $\langle r_i \rangle_{i=0}^{\infty}$ be an enumeration of the rational numbers in [0, 1) with $r_0 = 0$, and

define $P_i = P \overset{\circ}{+} r_i$. Then $P_0 = P$.

Claim: $P_i \cap P_j = \Phi$ if $i \neq j$.

Let $x \in P_i \cap P_j$. Then $x = p_i \overset{\circ}{+} r_i = p_j \overset{\circ}{+} r_j$ with p_i and p_j belonging to P. But $p_i - p_j = r_j - r_i$ is a rational number, whence $p_i \sim p_j$, i.e., p_i and p_j are in the same class. Since P has only one element from each equivalence class, we must have $i = j$. This implies that $P_i \cap P_j = \Phi$ if $i \neq j$.

i.e., $\langle P_i \rangle_{i=0}^{\infty}$ is a pair wise disjoint sequences of sets. On the other hand, each real number x in [0, 1) is in some equivalence class and so is equivalent to an element in P. But if x differs from an element in P by the rational number r_i, then $x \in P_i$. Thus $\bigcup P_i = [0, 1)$. Since each P_i is a translation modulo 1 of P, each P_i will be measurable if P is measuarble and will have the same measure i.e., $mP_i = mP$ for all i. But if this were the case,

$$m[0, 1) = \sum_{i=1}^{\infty} mP_i = \sum_{i=1}^{\infty} mP, \qquad (**)$$

and the right side is either zero or infinite, depending on whether mP is zero or positive. But this is impossible since $m[0, 1) = 1$, and consequently P cannot be measurable. **i.e., we have constructed a nonmeasurable set P.** We arrive at the following result:

(4.4) Theorem If m is a countably additive, translation invariant measure defined on a σ-algebra containing the set P seen above, then $m[0, 1)$ is either zero or infinite.

Proof If m is a countably additive, translation invariant measure defined on a σ-algebra containing the set P means measure of P is zero or positive. Now theorem follows from (**) in the construction (4.3).

---

[1]Axiom of choice: Let $\wp$ be any collection of nonempty sets. Then there is a function F defined on $\wp$ which assigns to each set $A \in \wp$ an element $F(A)$ in A.

The above equivalent to say that we have a set, say P, in $\wp$ which contains exactly one element from each set $A \in \wp$.

---

6 Measurable Functions

(5.1) Proposition Let f be an extended real-valued function whose domain is measurable. Then the following statements are equivalent:

i. For each real number α the set $\{x : f(x) > \alpha\}$ is measurable.

ii. For each real number α the set $\{x : f(x) \geq \alpha\}$ is measurable.

iii. For each real number α the set $\{x : f(x) < \alpha\}$ is measurable.

iv. For each real number α the set $\{x : f(x) \leq \alpha\}$ is measurable.

These statements imply

v. For each extended real number α the set $\{x: f(x) = \alpha\}$ is measurable.

Proof

Let the domain of f be D. Given that D is measurable.

(*i*) implies (*iv*): Note that $\{x: f(x) \le \alpha\} = D - \{x: f(x) > \alpha\}$.

D and $\{x: f(x) > \alpha\}$ are measurable implies the difference $D - \{x: f(x) > \alpha\}$ is also measurable. That is, $\{x: f(x) \le \alpha\}$ is measurable. That is, (*i*) implies (*iv*).

(*iv*) implies (*i*): Note that $\{x: f(x) > \alpha\} = D - \{x: f(x) \le \alpha\}$.

D and $\{x: f(x) \le \alpha\}$ are measurable implies the difference $D - \{x: f(x) \le \alpha\}$ is also measurable. That is, $\{x: f(x) > \alpha\}$ is measurable. That is, (*iv*) implies (*i*).

As above, **(*ii*) implies (*iii*)** and **(*iii*) implies (*ii*)** can be proved.

(*i*) implies (*ii*): Note that $\{x: f(x) \ge \alpha\} = \bigcap_{n=1}^{\infty} \left\{x: f(x) > \alpha - \frac{1}{n}\right\}$.

(*i*) implies that $\left\{x: f(x) > \alpha - \frac{1}{n}\right\}$ is measurable for any natural number n and, since intersection of a sequence of measurable sets is measurable, we have $\bigcap_{n=1}^{\infty} \left\{x: f(x) > \alpha - \frac{1}{n}\right\}$ is measurable. This implies that $\{x: f(x) \ge \alpha\}$ is measurable. That is, (*i*) implies (*ii*) is proved.

(*ii*) implies (*i*): Note that $\{x: f(x) > \alpha\} = \bigcup_{n=1}^{\infty} \left\{x: f(x) \ge \alpha + \frac{1}{n}\right\}$.

(*ii*) implies that $\left\{x: f(x) \ge \alpha + \frac{1}{n}\right\}$ is measurable for any natural number n and, since union of a sequence of measurable sets is measurable, we have $\bigcup_{n=1}^{\infty} \left\{x: f(x) \ge \alpha + \frac{1}{n}\right\}$ is measurable. This implies that $\{x: f(x) > \alpha\}$ is measurable. That is, (*ii*) implies (*i*) is proved.

That we have shown that the first four statements are equivalent.

Next we have to show that first four statements imply the fifth statement:

Case 1) If α is a real number:

$$\{x: f(x) = \alpha\} = \{x: f(x) \ge \alpha\} \cap \{x: f(x) \le \alpha\},$$

and so (*ii*) and (*iv*) implies (*v*)

Case 2) If $\alpha = \infty$:

$$\{x : f(x) = \infty\} = \bigcap_{n=1}^{\infty} \{x : f(x) \geq n\},$$

and so (*ii*) implies (*v*).

Case 3) If $\alpha = -\infty$:

$$\{x : f(x) = -\infty\} = \bigcap_{n=1}^{\infty} \{x : f(x) \leq -n\},$$

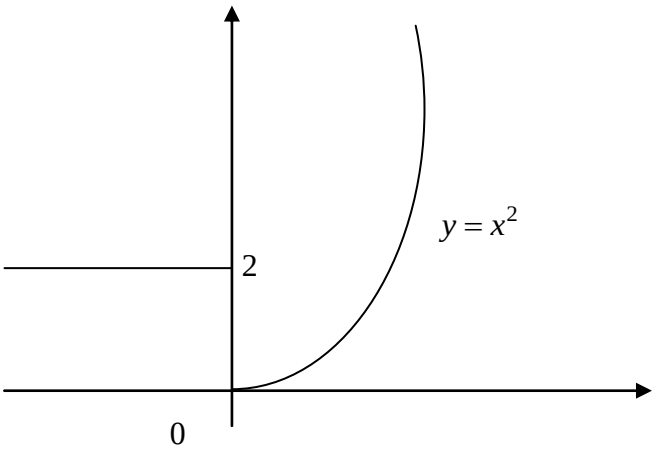
and so (*iv*) implies (*v*).

We have seen that, in any case (*ii*) and (*iv*) implies (*v*).

(5.2) Definition An extended real valued function f is said to be **(Lebesgue) measurable** if its domain is measurable and if it satisfies one of the first four statements of Proposition (5.1).

(5.3) Example Prove that $f(x) = \begin{cases} 2, & x \leq 0 \\ x^2, & x > 0 \end{cases}$ is measurable. (UQ 2006)

Solution



Graph of f

Case 1) $\alpha \leq 0$:

$\{x : f(x) \leq \alpha \leq 0\} = \Phi$ is measurable.

Case 2) $0 < \alpha \leq 2$:

$\{x : f(x) \leq \alpha\} = [0, \sqrt{\alpha}]$ is measurable.

Case 3) $\alpha > 2$:

$\{x : f(x) \leq \alpha\} = (-\infty, \sqrt{\alpha}]$ is measurable.

i.e., for any α, $\{x: f(x) \le \alpha\}$ is measurable. Hence f is measurable.

The following results (6.4) to (6.6) which are needed in the coming propositions are already known to the student and their proofs are omitted here.

(5.4) Proposition Every ordered field contains (sets isomorphic to) the natural numbers, the integers, and the rational numbers.

(5.5) Axiom of Archimedes Given any real number x, there is an integer n such that x $x < n$.

(5.6) Corollary to Axiom of Archimedes Between any two real numbers is a rational; that is, if $x < y$, then there is a rational r with $x < r < y$.

(5.7) Proposition Let c be a constant and f and g be two measurable real valued functions defined on the same domain. Then the functions $f + c$, cf, $f + g$, $g - f$, and fg are measurable.

Proof

(i) f and g are measurable implies their domain is measurable and it satisfies one of the first four statements of Proposition (5.1). We use condition (*iii*) of Proposition (5.1). Note that

$$\{x: (f + c)(x) < \alpha\} = \{x: f(x) + c < \alpha\} = \{x: f(x) < \alpha - c\}.$$

Since f is measurable, using condition (*iii*) of Proposition (5.1), we have $\{x: f(x) < \alpha - c\}$ is measurable. Hence $\{x: (f + c)(x) < \alpha\}$ is measurable. Hence the function $f + c$ is measurable.

(ii)

Case 1: For $c = 0$, $\{x: (cf)(x) < \alpha\} = \Phi$ for $\alpha \le 0$ and $\{x: (cf)(x) < \alpha\} =$ □ for $\alpha > 0$. In either case $\{x: (cf)(x) < \alpha\}$ is measurable since we know that Φ and □ are measurable sets.

Case 2: For positive c

$$\{x: (cf)(x) < \alpha\} = \{x: cf(x) < \alpha\} = \left\{x: f(x) < \frac{\alpha}{c}\right\}.$$

Since f is measurable, using condition (*iii*) of Proposition (5.1), we have $\left\{x: f(x) < \frac{\alpha}{c}\right\}$ is measurable. Hence $\{x:(cf)(x) < \alpha\}$ is measurable. Hence the function cf is measurable, for positive c.

Case 3: for negative c

$$\{x:(cf)(x) < \alpha\} = \{x: cf(x) < \alpha\} = \left\{x: f(x) > \frac{\alpha}{c}\right\}.$$

Since f is measurable, using condition (*i*) of Proposition (5.1), we have $\left\{x: f(x) > \frac{\alpha}{c}\right\}$ is measurable. Hence $\{x:(cf)(x) < \alpha\}$ is measurable. Hence the function cf is measurable, for negative c.

(iii) If $f(x) + g(x) < \alpha$, then $f(x) < \alpha - g(x)$ and by the Corollary (5.6) to the axiom of Archimedes there is a rational number r such that

$$f(x) < r < \alpha - g(x).$$

Hence

$$\{x: f(x) + g(x) < \alpha\} = \bigcap \{x: f(x) < r\} \bigcap \{x: g(x) < \alpha - r\}.$$

Since the rational are countable, this set if measurable and hence $f + g$ is measurable. Since $-g = (-1)g$ is measurable when g is, we have $f - g$ is measurable.

(iv) The function f^2 is measurable, since

Case 1) For $\alpha \geq 0$; $\{x: f^2(x) > \alpha\} = \{x: f(x) > \sqrt{\alpha}\} \cup \{x: f(x) < -\sqrt{\alpha}\}$ is measurable (being the union of two measurable sets)

and Case 2) For $\alpha < 0$, $\{x: f^2(x) > \alpha\} = D$ is measurable, since D the domain of f is measurable.

Also note that

$$fg = \frac{1}{2}\left[(f+g)^2 - f^2 - g^2\right] \tag{1}$$

In (1), RHS is measurable, as $f + g$ is measurable by (*i*) above and hence by the discussion just above $(f+g)^2$, f^2, g^2 are measurable and so is $\frac{1}{2}\left[(f+g)^2 - f^2 - g^2\right]$. Hence fg is measurable.

(5.8) Basic Definitions 1

Let $\langle x_n \rangle$ is a sequence of real numbers.

- The limit superior is defined by

$$\overline{\lim} x_n = \inf_n \sup_{k \geq n} x_k = \inf \{ \sup \{ x_k : k \geq n \} : n \in \mathbb{N} \}.$$

The symbols $\overline{\lim}$ and lim sup are both used for the limit superior.

- The limit inferior is defined by

$$\underline{\lim} x_n = \sup_n \inf_{k \geq n} x_k = \sup \{ \inf \{ x_k : k \geq n \} : n \in \mathbb{N} \}.$$

(5.9) Remark A real number l is the limit superior of the sequence $\langle x_n \rangle$ if and only if

(i) given $\varepsilon > 0$, there exists n such that $x_k < l + \varepsilon \ \forall k \geq n$, and

(ii) given $\varepsilon > 0$ and given n, there exists $k \geq n$ such that $l - \varepsilon < x_k$.

(5.10) Remarks

- The extended real number ∞ is the limit superior of the sequence $\langle x_n \rangle$ if and only if given Δ and n there exists $k \geq n$ such that $\Delta < x_k$.
- The extended real number $-\infty$ is the limit superior of the sequence $\langle x_n \rangle$ if and only if $-\infty = \lim x_n$.
- $\underline{\lim} x_n \leq \overline{\lim} x_n$.
- If $\langle x_n \rangle$ is a convergent sequence, then $\lim x_n = \underline{\lim} x_n = \overline{\lim} x_n$.

(5.11) Basic Definitions 2

Let $\langle f_n \rangle$ be a sequence of functions.

- The function $h = \sup \{ f_1, f_2, \ldots, f_n \}$ is defined by

$$h(x) = \sup \{ f_1(x), f_2(x), \ldots, f_n(x) \}.$$

- The function $g = \inf \{ f_1, f_2, \ldots, f_n \}$ is defined by

$$g(x) = \inf \{ f_1(x), f_2(x), \ldots, f_n(x) \}.$$

- The function $\sup_n f_n = \sup \{ f_1, f_2, \ldots, f_n, \ldots \}$ is defined by

$$\left(\sup_n f_n \right)(x) = \sup \{ f_1(x), f_2(x), \ldots, f_n(x), \ldots \}$$

- The function $\inf_n f_n = \inf\{f_1, f_2, \ldots, f_n, \ldots\}$ is defined by

$$\left(\inf_n f_n\right)(x) = \inf\{f_1(x), f_2(x), \ldots, f_n(x), \ldots\}$$

- $\overline{\lim} f_n$ is the limit superior function $\overline{\lim} f_n = \inf_n \sup_{k \geq n}\{f_k, f_{k+1}, \ldots\}$ defined by

$$\left(\overline{\lim} f_n\right)(x) = \inf_n \sup_{k \geq n}\{f_k(x), f_{k+1}(x), \ldots\} = \inf\left\{\sup\{f_k(x), f_{k+1}(x), \ldots, : k \geq n\} : n \in \square\right\}$$

$\overline{\lim}$ is also denoted by lim sup.

- $\underline{\lim} f_n$ is the limit inferior function $\underline{\lim} f_n = \sup_n \inf_{k \geq n}\{f_k, f_{k+1}, \ldots\}$ defined by

$$\underline{\lim} f_n(x) = \sup_n \inf_{k \geq n}\{f_k(x), f_{k+1}(x), \ldots\} = \sup\left\{\inf\{f_k(x), f_{k+1}(x), \ldots, : k \geq n\} : n \in \square\right\}$$

$\underline{\lim}$ is also denoted by lim inf.

(5.12) Theorem Let $\langle f_n \rangle$ be a sequence of measurable functions (with the same domain of definition). Then the functions $\sup\{f_1, \ldots, f_n\}$, $\inf\{f_1, \ldots, f_n\}$, $\sup_n f_n,,$ $\inf_n f_n,,$ $\overline{\lim} f_n$, and $\underline{\lim} f_n$ are all measurable.

Proof

(i) If h is defined by $h(x) = \sup\{f_1(x), \ldots, f_n(x)\}$, then

$$\{x : h(x) > \alpha\} = \bigcup_{i=1}^{n} \{x : f_i(x) > \alpha\}. \qquad (2)$$

As $f_1, \ldots, f_n$ are measurable, $\{x : f_i(x) > \alpha\}$ is measurable for $i = 1, \ldots, n.$ Being the union of measurable sets, $\bigcup_{i=1}^{n} \{x : f_i(x) > \alpha\}$ is measurable and hence $\{x : h(x) > \alpha\}$ is measurable and so h is measurable.

(ii) $\inf\{f_1, \ldots, f_n\}$ is measurable can be proved similarly.

(iii) If g is defined by $g(x) = \sup\{f_1(x), \ldots, f_n(x), \ldots\}$, then

$$\{x : g(x) > \alpha\} = \bigcup_{n=1}^{\infty} \{x : f_n(x) > \alpha\}. \qquad (2)$$

As $f_1, \ldots, f_n, \ldots$ are measurable, $\{x: f_n(x) > \alpha\}$ is measurable for $n = 1, 2, \ldots$. Being the countable union of measurable sets, $\bigcup_{n=1}^{\infty}\{x: f_n(x) > \alpha\} \bigcup_{i=1}^{n}\{x: f_i(x) > \alpha\}$ is measurable and hence $\{x: g(x) > \alpha\}$ is measurable and so g is measurable.

(iv) $\inf_n f_n = \inf\{f_1, f_2, \ldots, f_n, \ldots\}$ is measurable can be proved similarly.

(*v*) By the discussion above, inf function and sup function are measurable. Hence $\underline{\lim} f_n = \sup_n \inf_{k \geq n}\{f_k, f_{k+1}, \ldots,\}$ is measurable.

(*vi*) Similarly, $\overline{\lim} f_n = \inf_n \sup_{k \geq n}\{f_k, f_{k+1}, \ldots,\}$ is measurable.

(5.13) Definition A property is said to hold **almost everywhere** (abbreviated *a.e.*) if the set of points where it fails to hold is a set of measure zero.

- We say that $f = g$ a.e. if f and g have the same domain and $m\{x: f(x) \neq g(x)\} = 0$. (i.e., the measure of the set $\{x: f(x) \neq g(x)\}$ is 0.
- We say that f_n converges to g a.e. if there is a set E of measure zero such that $f_n(x)$ converges to $g(x)$ for each x not in E.

(5.14) Proposition If f is measurable function and $f = g$ a.e., then g is measurable.

Proof

Let $E = \{x: f(x) \neq g(x)\}$. Since $f = g$ a.e., we have $mE = 0$.

Now

$$\{x: g(x) > \alpha\} = \left[\{x: f(x) > \alpha\} \cup \{x \in E: g(x) > \alpha\}\right] \setminus \{x \in E: g(x) \leq \alpha\}.$$

The first set of the right is measurable, since f is a measurable function. The last two sets on the right are measurable, since they are subsets of E and $mE = 0$. Thus $\{x: g(x) > \alpha\}$ is measurable for each α, and so g is measurable.

(5.15) Proposition Let f be a measurable function defined on an interval $[a, b]$, and assume that f takes the values $\pm\infty$ only on a set of measure zero. Then given $\varepsilon > 0$, we can find a step function g and a continuous function h such that

$$|f - g| < \varepsilon \text{ and } |f - h| < \varepsilon$$

except on a set of measure less than ε; i.e., $m\{x:|f(x)-g(x)|\geq\varepsilon\}<\varepsilon$ and $m\{x:|f(x)-h(x)|\geq\varepsilon\}<\varepsilon.$ If in addition $m\leq f\leq M,$ then we may choose the functions g and h so that $m\leq g\leq M$ and $m\leq h\leq M.$

The above proposition tells us that a measurable function is "almost" a continuous function.

(5.16) Definition If A is any set, we define the **characteristic function** χ_A of the set A to be the function given by

$$\chi_A(x)=\begin{cases}1 \text{ if } x\in A\\ 0 \text{ if } x\notin A\end{cases}$$

The function χ_A is measurable if and only if A is measurable.

Thus the existence of a nonmeasurable set implies the existence of a nonmeasuarble function.

(5.17) Definition A real-valued function φ is called **simple** if it is measurable and assumes only a finite number of values.

If φ is simple and has the values $\alpha_1, \ldots, \alpha_n$ then

$$\varphi=\sum_{i=1}^{n}\alpha_i\chi_{A_i},$$

where $A_i=\{x:\varphi(x)=\alpha_i\}$. The sum, product, and difference of two simple functions are simple.

7 Littlewood's Three Principles

1. **Every (measurable) set is nearly a finite union of intervals.** Various forms of the first principle is given by Proposition (3.6)

2. **Every (measurable) function is nearly continuous.** One version of the second principle is given by Proposition (5.15).

3. **Every convergent sequence of (measurable) functions is nearly uniformly convergent.** One version of the third principle is given by the following Proposition.

(6.1) Proposition Let E be a measurable set of finite measure, and $\langle f_n \rangle$ a sequence of measurable functions defined on E. Let f be a real-valued function such that for each x in E we have $f_n(x) \to f(x)$. Then given $\varepsilon > 0$ and $\delta > 0$, there is a measurable set $A \subset E$ with $mA < \delta$ and an integer N such that for all $x \notin A$ and all $n \geq N$,

$$|f_n(x) - f(x)| < \varepsilon.$$

Proof

Let

$$G_n = \{x \in E : |f_n(x) - f(x)| \geq \varepsilon\},$$

and set

$$E_n = \bigcup_{n=N}^{\infty} G_n = \{x \in E : |f_n(x) - f(x)| \geq \varepsilon \text{ for some } n \geq N\}.$$

We have $E_{N+1} \subset E_N$, and for each $x \in E$ there must be some E_N to which x does not belong, since $f_n(x) \to f(x)$. Thus $\bigcap E_N = \Phi$, and so, by Proposition (3.4), $\lim mE_N = 0$. Hence given $\delta > 0$, there is N such that $mE_N < \delta$; that is,

$$m\{x \in E : |f_n(x) - f(x)| \geq \varepsilon \text{ for some } n \geq N\} < \delta.$$

If we write A for this E_N, then $mA < \delta$ and

$$\tilde{A} = \{x \in E : |f_n(x) - f(x)| < \varepsilon \text{ for all } n \geq N\}.$$

(6.2) Remark If, as in the hypothesis of the proposition, we have $f_n(x) \to f(x)$ for each x, we say that the sequence $\langle f_n \rangle$ converges **pointwise** to f on E. If there is a subset B of E with $mB = 0$ such that $f_n \to f$ pointwise on $E \setminus B$, we say that $f_n \to f$ a.e. on E. The following is the trivial modification of the last proposition:

(6.3) Proposition [Trivial modification of Proposition (6.1)] Let E be a measurable set of finite measure, and $\langle f_n \rangle$ a sequence of measurable functions that converge to a real-valued function f a.e. on E. Then, given $\varepsilon > 0$ and $\delta > 0$, there is a set $A \subset E$ with $mA < \delta$, and an N such that for all $x \notin A$ and all $n \geq N$,

$$|f_n(x) - f(x)| < \varepsilon.$$

Chapter 8

LEBESGUE INTEGRAL

The Riemann Integral

(8.1) The Riemann Integral

In this section we review Riemann integral using the notations that are used in this chapter.

Let f be a bounded real-valued function defined on the interval $[a, b]$ and let

$$a = \xi_0 < \xi_1 < \cdots < \xi_n = b$$

be a subdivision (some times called partition) of $[a, b]$. Then for each subdivision we can define the sums

$$S = \sum_{i=1}^{n} (\xi_i - \xi_{i-1}) M_i$$

and

$$s = \sum_{i=1}^{n} (\xi_i - \xi_{i-1}) m_i,$$

where $M_i = \sup_{\xi_{i-1} < x \le \xi_i} f(x), \qquad m_i = \inf_{\xi_{i-1} < x \le \xi_i} f(x).$

We define the **upper Riemann integral** of f by

$$\Re \bar{\int_a^b} f(x)dx = \inf S$$

with the infimum taken over all possible subdivisions (partitions) of $[a, b]$. Similarly, we define the **lower Riemann integral** of f by

$$\Re \underline{\int_a^b} f(x)dx = \sup s.$$

The upper integral is always at least as large as the lower integral.

If upper Riemann integral and lower Riemann integral of f are equal, then we say that f is **Riemann integrable** and the common value is the **Riemann integral** of f. The Riemann integral of f is denoted by

$$\Re \int_a^b f(x)dx$$

to distinguish it from the Lebesgue integral, which we consider in this chapter.

The upper and lower Riemann Integrals in terms of step functions

(8.2) Definition By a **step function** we mean a function ψ which has the form

$$\psi(x) = c_i, \qquad \xi_{i-1} < x < \xi_i$$

for some subdivision of $[a, b]$ and some set of constant c_i. Then the **integral of the step function** is given by

$$\int_a^b \psi(x)dx = \sum_{i=1}^{n} c_i \left(\xi_i - \xi_{i-1}\right).$$

(8.3) With the above in mind we see that the **upper Riemann integral of f in terms of the step functions** is

$$\Re \bar{\int_a^b} f(x)dx = \inf \int_a^b \psi(x)dx \tag{8.3}$$

for all step functions $\psi(x) \geq f(x)$.

(8.4) The **lower Riemann integral of f in terms of the step functions** is

$$\Re \underline{\int_a^b} f(x)dx = \sup \int_a^b \varphi(x)dx \tag{8.4}$$

for all step functions $\varphi(x) \leq f(x)$.

(8.5) Problem Show that if

$$f(x) = \begin{cases} 0 & x \text{ irrational} \\ 0 & x \text{ rational} \end{cases},$$

then $$\Re \bar{\int_a^b} f(x)dx = b - a \quad \text{and} \quad \Re \underline{\int_a^b} f(x)dx = 0.$$

[Hint: For the step functions $\psi(x) = \begin{cases} 1, & a \leq x \leq b \\ 0, & \text{elsewhere} \end{cases}$ and $\varphi(x) = 0 \ \forall x$, we have $\varphi(x) \leq f(x) \leq \psi(x)$.]

The Lebesgue Integral of a Bounded Function over a Set of Finite Measure

(8.6) Definition The function χ_E defined by

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$

is called the **characteristic function of *E*.**

(8.7) Definition A linear combination

$$\varphi(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x) \tag{1}$$

is called a **simple function** if the sets E_i are measurable. This representation for φ is not unique. However, we note that a function φ is simple if and only if it is measurable and assumes only a finite number of values.

(8.8) Definition If φ is a simple function and $\{a_1, \ldots, a_n\}$ the set of nonzero values of φ, then

$$\varphi = \sum_{i=1}^{n} a_i \chi_{A_i},$$

where $$A_i = \{x : \varphi(x) = a_i\}.$$

This representation for φ is called the **canonical representation**, and it is characterized by the fact that the A_i are disjoint and the a_i distinct and nonzero.

(8.9) Definition *(Integral of a simple function)* If φ is a simple function that vanishes outside a set of finite measure, we define the **integral of** φ by

$$\int \varphi(x)dx = \sum_{i=1}^{n} a_i m A_i$$

when φ has the canonical representation $\varphi = \sum_{i=1}^{n} a_i \chi_{A_i}$.

(8.10) Remark Read and understand the underlined words because now the integral of the simple function is given by means of its *canonical representation* (We will see in Remark (8.13) that this is not necessary). Now, by the Definition (8.9), the integral of the simple function φ cannot be determined immediately using (1) of Definition (8.7); to evaluate the

integral we have first to write φ in some canonical representation as in Definition (8.8). This will be clear to the student from the proof of the coming Lemma (8.11).

We sometimes abbreviate the expression for this integral to $\int\varphi$. If E is any measurable set, we define

$$\int_E \varphi = \int \varphi \cdot \chi_E .$$

(8.11) Lemma Let $\varphi = \sum_{i=1}^{n} a_i \chi_{E_i}$, with $E_i \cap E_j = \Phi$ for $i \neq j$. Suppose each set E_i is a measurable set of finite measure. Then

$$\int \varphi = \sum_{i=1}^{n} a_i m E_i . \qquad \text{(UQ 2006)}$$

Proof

To determine $\int\varphi$ using Definition (8.9) we have first to find the canonical representation of φ [Read Remark (8.10)]

To determine the canonical representation, we consider the sets

$$A_a = \{x : \varphi(x) = a\} = \bigcup_{a_i = a} E_i . \qquad \text{[Ref. the following figure]}$$

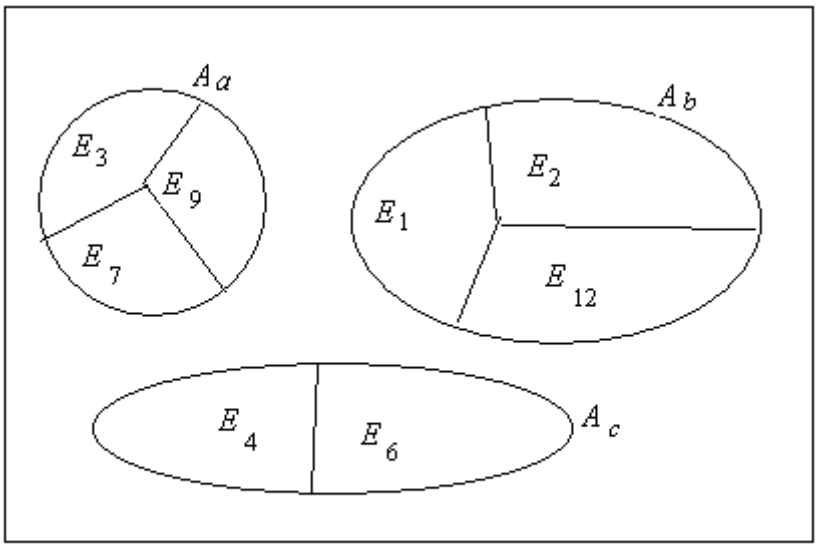


Figure: Venn Diagram showing A_a, A_b, A_c

The Venn Diagram is drawn as for $x \in E_3$, $\varphi(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x) = a_3$, [since $\chi_{E_i}(x) = \begin{cases} 1 & i = 3 \\ 0 & i \neq 3 \end{cases}$] and assuming that $a_3 = a$. Similarly, assuming $a_7 = a$ and $a_9 = a$, we have $\varphi(x) = a$ for $x \in E_7, E_9$; Similarly, $\varphi(x) = b$ for $x \in E_1, E_2, E_{12}$; $\varphi(x) = c$ for $x \in E_4, E_6$.

Then the canonical representation of φ is

$$\varphi = \sum a \chi_{A_a}.$$

Hence by Definition (8.9),

$$\int \varphi(x)dx = \sum amA_a \tag{2}$$

Since $\{E_i\}$ is pairwise disjoint, by the additivity of m, we have

$$amA_a = \sum_{a_i = a} a_i mE_i, \tag{3}$$

Hence from (2), we have

$$\int \varphi(x)dx = \sum a_i mE_i.$$

(8.12) Proposition Let φ and ψ be simple functions which vanish outside a set of finite measure. Then

(*a*) $$\int (a\varphi + b\psi) = a\int \varphi + b\int \psi,$$

and

(*b*) if $\varphi \geq \psi$ a.e., then

$$\int \varphi \geq \int \psi.$$

Proof

Let $\varphi = \sum_{i=1}^{n} a_i \chi_{A_i}$, where $A_i = \{x : \varphi(x) = a_i\}$ and $\psi = \sum_{i=1}^{m} b_i \chi_{B_i}$, where $B_i = \{x : \varphi(x) = b_i\}$ be the canonical representations of φ and ψ, respectively. Then $\{A_i\}$ and $\{B_i\}$ be the sets occurring in canonical representations of φ and ψ.

Let A_0 and B_0 be the sets where φ and ψ are zero.

We consider the intersections $A_i \cap B_j$. These intersections form a finite disjoint collection of measurable sets and we denote the collection by $\{E_k\}_{k=1}^{n}$.

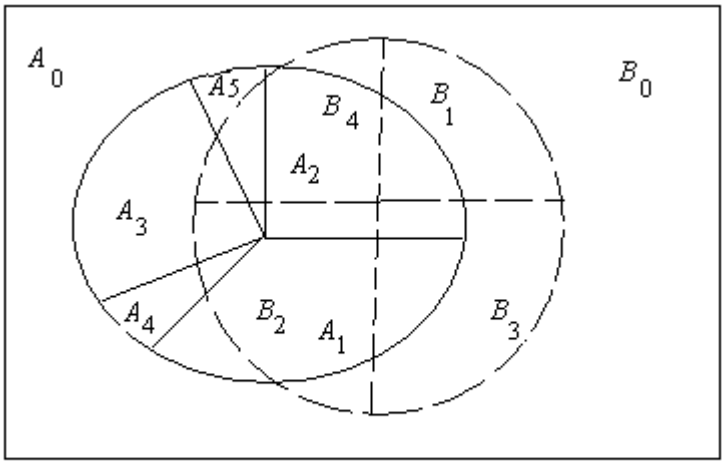


Figure: Venn Diagram for {Ei}
Dashed curves for B's and the other for A's

Then

$$\varphi = \sum_{k=1}^{N} a_k \chi_{E_k} \text{ , with } E_i \cap E_j = \Phi, \text{ for } i \neq j. \tag{4}$$

$$\psi = \sum_{k=1}^{N} b_k \chi_{E_k} \text{ , with } E_i \cap E_j = \Phi, \text{ for } i \neq j. \tag{5}$$

(4) and (5) need not be the canonical representations of φ and ψ. But we are now in a position to use Lemma (8.11), since $E_i \cap E_j = \Phi, \text{ for } i \neq j.$

i.e., by Lemma (8.11), we have

$$\int \varphi = \sum_{k=1}^{N} a_k m E_k$$

and

$$\int \psi = \sum_{k=1}^{N} b_k m E_k \text{ .}$$

Hence

$$a\int \varphi + b\int \psi = a\sum_{k=1}^{N} a_k m E_k + b\sum_{k=1}^{N} b_k m E_k = \sum_{k=1}^{N} \left(a a_k m E_k + b b_k m E_k \right) \tag{6}$$

Also, from (4) and (5)

$$a\varphi + b\psi = \sum_{k=1}^{N} \left(a a_k + b b_k \right) \chi_{E_k} \text{ ,}$$

whence, again by Lemma (8.11) , it follows that

$$\int (a\varphi + b\psi) = \sum_{k=1}^{N} (aa_k + bb_k) mE_k. \tag{7}$$

R.H.S. of (6) and (7) are the same. Hence,

$$\int (a\varphi + b\psi) = a\int \varphi + b\int \psi.$$

Proof of the second statement:

$\varphi \geq \psi$ a.e. implies $\varphi - \psi \geq 0$ a.e. implies $\varphi - \psi$ is a non-negative simple function. Then if the canonical representation of $\varphi - \psi$ is $\varphi = \sum_{i=1}^{n} c_i \chi_{C_i}$, where $C_i = \{x : \varphi(x) = c_i\}$, then $\varphi - \psi$ is a non-negative implies $c_i \geq 0$.

Also, by Definition (8.9),

$$\int \varphi - \psi = \sum_{i=1}^{n} c_i mC_i \ . \tag{8}$$

Hence by noting that $c_i \geq 0$ and m is a nonnegative function the above implies

$$\int \varphi - \psi \geq 0. \tag{9}$$

Taking $a = 1, b = -1,$ part (*a*) implies

$$\int \varphi - \int \psi = \int \varphi - \psi.$$

Hence by (9), we have

$$\int \varphi - \int \psi \geq 0.$$

i.e., $$\int \varphi \geq \int \psi.$$

(8.13) Remark Putting $a = 1, b = 0$ in (7) of the proof above, it follows that, if $\varphi = \sum_{i=1}^{n} a_i \chi_{E_i}$, then $\int \varphi = \sum_{i=1}^{n} a_i mE_i$, and so the restriction of Lemma (8.11) that the sets E_i be disjoint is unnecessary.

Let f be a bounded real valued function and E a measurable set of finite measure. By analogy with the Riemann integral we consider for simple functions φ and ψ the numbers

$$\inf \left\{ \int_E \psi : \psi \geq f \right\} \qquad \text{(usually denoted by } \inf_{\psi \geq f} \int_E \psi \text{)}$$

and $\sup\left\{\int_E \varphi : \varphi \le f\right\}$, (usually denoted by $\sup_{\varphi \le f}\int_E \varphi$)

and ask when these two numbers are equal. The answer is given by the following proposition:

(8.14) Proposition Let f be defined and bounded on a measurable set E with mE finite. In order that

$$\inf_{f\le\psi}\int_E \psi(x)\,dx = \sup_{f\ge\varphi}\int_E \varphi(x)dx$$

for all simple functions φ and ψ, it is necessary and sufficient that f be measurable.

Proof Let f be bounded by M and suppose that f is measurable.

f be bounded by M implies $|f(x)| \le M$ implies $-M \le f(x) \le M \quad \forall x$

or $M \ge f(x) \ge -M \quad \forall x.$

Then we consider the sets

$$E_k = \left\{x : \frac{kM}{n} \ge f(x) \ge \frac{(k-1)M}{n}\right\} \qquad -n \le k \le n,$$

that are measurable, disjoint, and have union E.

Some particular sets E_k :

For $k = n$, $E_n = \left\{x : M \ge f(x) \ge \frac{(n-1)}{n}M\right\}$; For $k = -n$, $E_{-n} = \left\{x : -M \ge f(x) \ge \frac{-(n+1)}{n}M\right\}$

Since E_k are disjoint and have union E, we have

$$\sum_{k=-n}^{n} mE_k = mE.$$

The simple functions defined by

$$\psi_n(x) = \frac{M}{n}\sum_{k=-n}^{n} k\chi_{E_k(x)}$$

and

$$\varphi_n(x) = \frac{M}{n}\sum_{k=-n}^{n} (k-1)\chi_{E_k(x)}$$

satisfy

$$\varphi_n(x) \le f(x) \le \psi_n(x).$$

Thus

$$\inf_{f\le\psi}\int_E \psi(x)\,dx \overset{*}{\le} \int_E \psi_n(x)dx \overset{*}{=} \frac{M}{n}\sum_{k=-n}^{n} kmE_k$$

The reason for the inequality "$\overset{*}{\leq}$": $\inf\limits_{f\leq\psi}\int_E \psi(x)\,dx = \inf\left\{\int_E \psi(x)\,dx : f\leq\psi,\ \psi \text{ is simple}\right\}$ and $\int_E \psi_n(x)dx \in \left\{\int_E \psi(x)\,dx : f\leq\psi,\ \psi \text{ is simple}\right\}$. The equality "$\overset{*}{=}$" follows from Remark (8.13)

.

and

$$\sup_{f\geq\varphi}\int_E \varphi(x)dx \geq \int_E \varphi_n(x)dx = \frac{M}{n}\sum_{k=-n}^{n}(k-1)mE_k,$$

whence

$$0 \leq \inf_{f\leq\psi}\int_E \psi(x)\,dx - \sup_{f\geq\varphi}\int_E \varphi(x)dx \leq \frac{M}{n}\sum_{k=-n}^{n} mE_k = \frac{M}{n}mE.$$

Since n is arbitrary, we have

$$0 \leq \inf_{f\leq\psi}\int_E \psi(x)\,dx - \sup_{f\geq\varphi}\int_E \varphi(x)dx \leq 0.$$

Hence

$$\inf_{f\leq\psi}\int_E \psi(x)\,dx - \sup_{f\geq\varphi}\int_E \varphi(x)dx = 0$$

implies

$$\inf_{f\leq\psi}\int_E \psi(x)\,dx = \sup_{f\geq\varphi}\int_E \varphi(x)dx$$

implies the condition is sufficient.

Suppose now that φ_n and

$$\inf_{f\leq\psi}\int_E \psi(x)\,dx = \sup_{f\geq\varphi}\int_E \varphi(x)dx = l.$$

Then, given n, $l+\frac{1}{n}$ cannot be a lower bound implies there is a simple function ψ_n such that $\psi_n(x)\geq f(x)$ and $\int_E \psi_n(x)\,dx < l+\frac{1}{2n}$. Similarly, $l-\frac{1}{n}$ cannot be an upper bound and this implies there is a simple function φ_n such that $f(x)\geq\varphi_n(x)$ and $\int_E \varphi_n(x)\,dx > l-\frac{1}{2n}$. Now $-l-\frac{1}{2n} < -\int_E \psi_n(x)\,dx$ and $l-\frac{1}{2n} < \int_E \varphi_n(x)\,dx$ implies $-\frac{1}{n} < -\int_E \psi_n(x)dx + \int_E \varphi_n(x)dx.$

i.e., for given n, there are simple functions φ_n and ψ_n such that

$$\varphi_n(x) \le f(x) \le \psi_n(x)$$

and

$$\int_E \psi_n(x)dx - \int_E \varphi_n(x)dx < \frac{1}{n}.$$

Note that being the simple functions, ψ_n's and φ_n's are measurable. Hence by Theorem (5.12) of the previous chapter, we have the functions

$$\psi^* = \inf_n \psi_n$$

and $$\varphi^* = \sup_n \varphi_n$$

are measurable, and

$$\varphi^*(x) \le f(x) \le \psi^*(x). \tag{10}$$

Now the set

$$\Delta = \left\{x : \varphi^*(x) < \psi^*(x)\right\}$$

is the union of the sets

$$\Delta_v = \left\{x : \varphi^*(x) < \psi^*(x) - \frac{1}{v}\right\}.$$

But each Δ_v is contained in the set $\left\{x : \varphi_n(x) < \psi_n(x) - \frac{1}{v}\right\}$, and this latter set has measure less than $\frac{v}{n}$. Since n is arbitrary, $m\Delta_v = 0$, and so $m\Delta = 0$. Thus $\varphi^* = \psi^*$ except on a set of measure zero, and hence by (10) $\varphi^* = f$ $\left(= \psi^*\right)$ except on a set of measure zero. Thus f is measurable by Proposition (5.14) of the previous chapter, and the condition is also necessary.

(8.15) Definition *(Integral of a Bounded Measurable Function)*If f be a <u>bounded</u> measurable function defined on a measurable set E with mE finite, we define the (**Lebesgue**) **integral** of f over E by

$$\int_E f(x)dx = \inf_{\psi \ge f} \int_E \psi(x)dx$$

for all simple functions $\psi \ge f$.

i.e., $$\int_E f(x)dx = \inf\left\{\int_E \psi(x)dx : \psi \text{ is simple function and } \psi \ge f\right\}.$$

(8.16) Notation We sometimes write the integral as $\int_E f$. If $E=[a, b]$, we write $\int_a^b f$ instead of $\int_{[a,b]} f$. If f is a bounded measurable function that vanishes outside a set E of finite measure, we write $\int f$ for $\int_E f$. Note that $\int_E f = \int_E f \cdot \chi_E$.

(8.17) Proposition (Corollary of Proposition (8.14))

Let f be a bounded function defined on $[a,b]$. If f is a Riemann integral on $[a,b]$, then it is measurable and

$$\Re\int_a^b f(x)dx = \int_a^b f(x)dx.$$

i.e., Lebesgue integral is in fact a generalization of a the Riemann integral.

Proof

By (8.4),

$$\Re\int_{_a}^b f(x)dx = \sup\int_a^b \varphi(x)dx$$

for all step functions $\varphi(x) \le f(x)$. Since every step function is also a simple function, we have for all simple functions $\varphi_1 \le f$

$$\sup_{\varphi\le f}\int_a^b \varphi(x)dx \le \sup_{\varphi_1\le f}\int_a^b \varphi_1(x)dx\,.$$

Hence for all simple functions $\varphi_1 \le f$

$$\Re\int_{_a}^b f(x)dx \le \sup_{\varphi_1\le f}\int_a^b \varphi_1(x)dx \tag{8.5}$$

Similarly, considering (8.3), for all simple functions $\psi_1 \ge f$

$$\inf_{\psi_1\ge f}\int_a^b \psi_1(x)dx \le \Re\int_a^{\bar{b}} f(x)dx. \tag{8.6}$$

But,
$$\sup_{\varphi_1\le f}\int_a^b \varphi_1(x)dx \le \inf_{\psi_1\ge f}\int_a^b \psi_1(x)dx \tag{8.7}$$

Combining (8.5) to (8.7), we obtain

$$\Re\int_{_a}^{b} f(x)dx \le \sup_{\varphi_1 \le f}\int_a^b \varphi_1(x)dx \le \inf_{\psi_1 \ge f}\int_a^b \psi_1(x)dx \le \Re\bar{\int_a^b} f(x)dx. \qquad (8.8)$$

If f is a Riemann integral on $[a,b]$, then $\Re\int_{_a}^{b} f(x)dx = \Re\bar{\int_a^b} f(x)dx$, so from (8.8), we obtain

$$\sup_{\varphi_1 \le f}\int_a^b \varphi_1(x)dx = \inf_{\psi_1 \ge f}\int_a^b \psi_1(x)dx$$

and hence by Proposition (8.14) f is measurable. Also, using the Definition (8.15), we have

$$\Re\int_a^b f(x)dx = \int_a^b f(x)dx.$$

(8.18) Proposition If f and g are bounded measurable functions defined on a set E of finite measure, then:

i. $\int_E (af + bg) = a\int_E f + b\int_E g.$

ii. If $f = g$ a.e., then

$$\int_E f = \int_E g.$$

iii. (a) If $f \le g$ a.e., then

$$\int_E f \le \int_E g.$$

(b) Hence $$\left|\int f\right| \le \int |f|.$$

iv. If $A \le f(x) \le B,$ then

$$AmE \le \int_E f \le BmE.$$

v. If A and $\mathcal{B}$ are disjoint measurable sets of finite measure, then

$$\int_{A\cup B} f \le \int_A f + \int_B f.$$

Proof

(*i*) Note that if ψ is a simple function so is $a\psi$, and conversely if $a \neq 0$, then $a\psi$ is simple.

For $a = 0$ there is nothing to prove.

For $a > 0$,

$$\int_E af = \inf_{a\psi \geq af} \int_E (a\psi)(x)dx, \text{ by Definition (8.15)}$$

$$= \inf_{\psi \geq f} \int_E a\psi(x)dx = a \inf_{\psi \geq f} \int_E \psi(x)dx = a\int_E f.$$

If $a < 0$,

$$\int_E af = \inf_{a\psi \geq af} \int_E (a\psi)(x)dx, \text{ using Definition (8.15)}$$

$$= \inf_{\substack{\psi \leq f \\ \uparrow}} \int_E a\psi(x)dx, \text{ since } a < 0$$

$$= a \sup_{\psi \leq f}^{\downarrow} \int_E \psi(x)dx \text{ , since } a < 0$$

$$= a \inf_{\substack{\psi \geq f \\ \uparrow\uparrow}}^{\downarrow\downarrow} \int_E \psi(x)dx, \text{ since } \psi \text{ is measurable and using Proposition (8.14)}$$

$$= a\int_E f \text{ , by Definition (8.15)} \qquad (1)$$

[In the above $\uparrow$ and $\uparrow\uparrow$ are used to indicate the change in the inequality. $\downarrow$ and $\downarrow\downarrow$ are used to indicate the change of inf and sup].

If ψ_1 is a simple function $f \leq \psi_1$, and ψ_2 a simple function, $g \leq \psi_2$, then $\psi_1 + \psi_2$ is a simple function $f + g \leq \psi_1 + \psi_2$. Hence

$$\int_E f + g \leq \int_E (\psi_1 + \psi_2), \qquad (2)$$

since by definition $\int_E f + g = \inf_{f+g \leq \psi} \int_E \psi$.

Also, by (*a*) of Proposition (8.12),

$$\int_E (\psi_1 + \psi_2) \square \int_E \psi_1 + \int_E \psi_2 .$$

Hence (2) becomes

$$\int_E f + g \le \int_E \psi_1 + \int_E \psi_2. \tag{3}$$

Since the infimum on the right side is $\int f + \int g,$ we have

$$\int_E f + g \le \int_E f + \int_E g. \tag{4}$$

On the other hand, $\varphi_1 \le f$ and $\varphi_2 \le g$ imply $\varphi_1 + \varphi_2$ is a simple function not greater than $f + g.$ Hence

$$\int_E f + g \ge \int_E (\varphi_1 + \varphi_2), \tag{5}$$

since $\int_E f + g = \sup_{\varphi \le f+g} \int_E \varphi.$

Again by Proposition (8.12),

$$\int_E (\varphi_1 + \varphi_2) = \int_E \varphi_1 + \int_E \varphi_2.$$

Hence (5) becomes

$$\int_E f + g \ge \int_E \varphi_1 + \int_E \varphi_2. \tag{6}$$

Since now the supremum of the right side is $\int f + \int g,$ we have

$$\int_E f + g \ge \int_E f + \int_E g, \tag{7}$$

hence, (4) and (7) gives

$$\int_E f + g = \int_E f + \int_E g. \tag{8}$$

Using (1) and (8) it follows that

$$\int_E (af + bg) = \int_E (af) + \int_E (bg) = a\int_E f + b\int_E g.$$

(*ii*) To prove (*ii*), it now suffices to show that

$$\int_E f - g = 0.$$

Since $f - g = 0$ a.e., it follows that if $\psi \geq f - g,\ \psi \geq 0$ a.e. From this it follows [referring to (9) of Proposition (8.12)] that

$$\int_E \psi \geq 0,$$

whence

$$\int_E f - g = \inf_{\psi \geq f-g} \int_E \psi \ \geq\ 0.$$

Similarly, if $\varphi \leq f - g,\ \varphi \leq 0$ a.e. and $\int_E \varphi \leq 0,$ and hence

$$\int_E f - g = \sup_{\varphi \leq f-g} \int_E \varphi \ \leq 0,$$

whence (*ii*) is proved.

(*iii*) (a) The above proof also serves to establish (*iii*). [Details: $f \leq g$ a.e. implies $f - g \leq 0$ a.e. If $\varphi \leq f - g,\ \varphi \leq 0$ a.e. and $\int_E \varphi \leq 0,$ and hence

$$\int_E f - g = \sup_{\varphi \leq f-g} \int_E \varphi \ \leq 0, \tag{9}$$

and by (*i*) $\int_E f - g = \int_E f - \int_E g$, so (9) gives

$$\int_E f - \int_E g \leq 0$$

i.e.,

$$\int_E f \leq \int_E g.$$

(*iii*) (b) Now, $f \leq |f|$ a.e. implies [by (*iii*)(a)] that

$$\int_E f \leq \int_E |f|. \tag{10}$$

$-f \leq |f|$ a.e. implies [by (*iii*)(a)] that $\int_E -f \leq \int_E |f|.$

Also, by (*i*) $\int_E -f = -\int_E f$. Hence, we have

$$-\int_E f \le \int_E |f|. \tag{11}$$

From (10) and (11), we have $\left|\int f\right| \le \int |f|$.

(iv)

$A \le f(x)$ implies, by (*iii*)(*a*), $\int_E A \le \int_E f$ implies, by (*i*), $A\int_E 1 \le \int_E f$ implies $AmE \le \int_E f$, since $\int_E 1 = mE$. Similarly, $f(x) \le B$ implies, by (*iii*)(*a*), $\int_E f \le \int_E B$ implies, by (*i*), $\int_E f \le B\int_E 1$ implies $\int_E f \le BmE$. This proves (*iv*).

(v)

$$\int_{A\cup B} f = \int_{A\cup B} f\chi_{A\cup B}$$

$$\le \int_{A\cup B} f(\chi_A + \chi_B), \text{ since } \chi_{A\cup B} = \chi_A + \chi_B \text{ when } A \text{ and } B \text{ are disjoint.}$$

$$= \int_{A\cup B} f\chi_A + f\chi_B$$

$$= \int_{A\cup B} f\chi_A + \int_{A\cup B} f\chi_B \text{, using (}i\text{)}$$

$$= \int_A f\chi_A + \int_B f\chi_B$$

$$= \int_A f + \int_B f.$$

(8.19) Proposition (Bounded Convergence Theorem) Let $\langle f_n \rangle$ be a sequence of measurable functions defined on a set E of finite measure and suppose that there is a real number M such that $|f_n(x)| \le M$ for all n and all x. If $f(x) = \lim f_n(x)$ for each x in E, then

$$\int_E f = \lim \int_E f_n.$$

Proof

The proof of this propostion furnishes a nice illustration of the use of Littlewood's "three principles". The construction of the propostion would be trivial if $\langle f_n \rangle$ converged to f uniformly, Littlewood's three principles states that if $\langle f_n \rangle$ converges to f pointwise, then $\langle f_n \rangle$ is "nearly" uniformly convergent to f. A precise version of this principle is given by Propsotion in the previous chapter, which states that, given $\varepsilon > 0$, there is an N and a measurable set $A \subset E$ with $mA < \frac{\varepsilon}{4M}$ such that for $n \geq N$ and $x \in E \setminus A$ we have

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2mE}.$$

Thus

$$\left| \int_E f_n - \int_E f \right| = \left| \int_E f_n - f \right|, \text{ by } (i) \text{ of Proposition (8.18)}$$

$$\leq \int_E |f_n - f| \text{ , by } (iii) \text{ of Proposition (8.18)}$$

$$= \int_{E \setminus A} |f_n - f| + \int_A |f_n - f|$$

$$< \frac{\varepsilon}{2mE} \int_{E \setminus A} 1 + 2\varepsilon M \int_A 1$$

$$< \frac{\varepsilon}{2mE} mE + 2\varepsilon M \frac{\varepsilon}{4M}, \text{ since } \int_{E \setminus A} 1 = m(E \setminus A) \leq mE$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\int_E f = \lim \int_E f_n.$$

(8.20) Proposition A bounded function f on $[a,b]$ is Riemann integrable, if and only if the set of points at which f is discontinuous has measure zero.

Hint of the proof

Let f be a bounded function on $[a, b]$, and let h be the upper envelope of f. Then show that $\mathfrak{R}\int_a^{\bar{b}} f(x)dx = \int_a^b h$. **[** If $\varphi \geq f$ is a step function, then $\varphi \geq h$ except at a finite number of points, and so $\int_a^b h \leq \mathfrak{R}\int_a^{\bar{b}} f(x)dx$. But there is a sequence $\langle \varphi_n \rangle$ of step functions such that $\varphi_n \downarrow h$. By Proposition (8.19), we have $\int_a^b h = \lim \int_a^b \varphi_n \geq \mathfrak{R}\int_a^{\bar{b}} f(x)dx$.**]** Use this to prove Proposition (8.20).

2 The Integral of a Nonnegative Function

(8.21) Definition *(Integral of a Nonnegative Measurable Function)* If f is a nonnegative measurable function defined on a measurable set E, we define

$$\int_E f = \sup_{h \leq f} \int_E h,$$

where h is a bounded measurable function such that $m\{x : h(x) \neq 0\}$ is finite.

(8.22) Proposition If f and g are nonnegative measurable functions, then:

i. $$\int_E cf = c\int_E f, \qquad c > 0.$$

ii. $$\int_E f + g = \int_E f + \int_E g.$$

iii. If $f \leq g$ a.e., then

$$\int_E f \leq \int_E g.$$

Proof

(*i*)

For $c > 0$,

$$\int_E cf = \sup_{ch \leq cf} \int_E ch,$$ where ch is a bounded measurable function such that $m\{x : (ch)(x) \neq 0\}$

$$= \sup_{h \le f} \int_E ch$$

$$= \sup_{h \le f} c \int_E h \text{, by } (i) \text{ of Proposition (8.18)}$$

$$= c \sup_{h \le f} \int_E h$$

$$= c \int_E f.$$

For $c < 0$,

$$\int_E cf = \sup_{ch \le cf} \int_E ch$$

$$= \sup_{\substack{h \ge f \\ \uparrow}} \int_E ch$$

$$= \sup_{h \ge f} c \int_E h \text{, by Proposition (8.18)}$$

$$= c \inf_{h \ge f} \int_E h \text{, since } c < 0,$$

$$= c \sup_{g \le f} \int_E h, \quad \text{since} \quad \inf_{h \ge f} \int_E h = \sup_{g \le f} \int_E g \quad \text{for} \quad \underline{\text{bounded}} \quad \text{measurable functions } g \text{ and } h.$$

$$= c \int_E f.$$

(*ii*)

If $h(x) \le f(x)$ and $k(x) \le g(x)$, we have $h(x) + k(x) \le f(x) + g(x)$, and so

$$\int_E h + \int_E k \le \int_E f + g.$$

Taking suprema, we have

$$\int_E f + \int_E g \le \int_E f + g.$$

On the other hand, let l be a bounded measurable function which vanishes outside a set of finite measure and which is not greater than $f + g$. Then we define the functions h and k by setting

$$h(x) = \min\big(f(x), l(x)\big)$$

and $k(x) = l(x) - h(x).$

We have

$$h(x) \le f(x)$$

and $k(x) \le g(x),$

while h and k are bounded by the bound for l and vanish where l vanishes. Hence $\int_E l = \int_E h + \int_E h \le \int_E f + \int_E g,$ and so

$$\int_E f + \int_E g \ge \int_E f + g.$$

Hence (*ii*) is proved.

Part (*iii*) follow directly from Proposition (8.18).

(8.23) Theorem (Fatou's Lemma) If $\langle f_n \rangle$ is a sequence of nonnegative measurable functions and $f_n(x) \to f(x)$ almost everywhere on a set E, then

$$\int_E f \le \underline{\lim} \int_E f_n.$$

Proof

Without loss of generality, we may assume that the convergence is everywhere, since integrals over sets of measure zero are zero. Let h be a bounded measurable function which is not greater than f and which vanishes outside a set E' of finite measure. Define a function h_n by setting

$$h_n(x) = \min\big(h(x), f_n(x)\big)$$

Then h_n is bounded by the bound (say M) for h and vanishes outside E'. Then $|h_n(x)| \le M$

Also $h_n(x) \to h(x)$ for each x in E'. Thus we have

$$\int_E h = \int_{E'} h = \lim \int_{E'} h_n \quad \text{by Bounded Convergence Theorem (Proposition (8.19)),}$$

$$\le \underline{\lim} \int_E f_n.$$

i.e., $$\int_E h \le \underline{\lim} \int_E f_n.$$

Taking the supremum over h, we get (using Definition (8.21))

$$\int_E f \leq \underline{\lim} \int_E f_n.$$

This completes the proof.

(8.24) Monotone Convergence Theorem Let $\langle f_n \rangle$ be an increasing sequence of nonnegative measurable functions, and let $f = \lim f_n$ a.e. Then

$$\int f = \lim \int f_n.$$

Proof

By Fatou's Lemma , we have

$$\int f \leq \underline{\lim} \int f_n. \tag{1}$$

But for each n we have $f_n \leq f$, and so $\int f_n \leq \int f$. But this implies

$$\overline{\lim} \int f_n \leq \int f \ . \tag{2}$$

Since $\underline{\lim} \int f_n \leq \overline{\lim} \int f_n$, (1) and (2) gives

$$\int f \leq \underline{\lim} \int f_n \leq \overline{\lim} \int f_n \leq \int f \ . \tag{3}$$

(3) tells us that all the inequalities are equal and hence

$$\underline{\lim} \int f_n = \overline{\lim} \int f_n = \int f,$$

and hence $\lim \int f_n$ exists and

$$\lim \int f_n = \int f.$$

(8.25) Corollary Let u_n be a sequence of nonnegative measurable functions, and let $f = \sum_{n=1}^{\infty} u_n$. Then

$$\int f = \sum_{n=1}^{\infty} \int u_n .$$

i.e., $$\int \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \int u_n .$$

Proof Take $f_n = \sum_{k=1}^{n} u_k$. Since each u_k is nonnegative, $\langle f_n \rangle$ is an increasing sequence of nonnegative measurable functions, and $f = \lim f_n$. Then by Monotone Convergence Theorem (8.24),

$$\int f = \lim \int f_n$$

$$= \lim \int \sum_{k=1}^{n} u_k$$

$$= \lim \sum_{k=1}^{n} \int u_k \text{ , using Proposition (8.22)}$$

$$= \sum_{n=1}^{\infty} \int u_n \ .$$

(8.26) Proposition Let f be a nonnegative function and $\langle E_i \rangle$ a disjoint sequence of measurable sets. Let $E = \bigcup E_i$. Then

$$\int_E f = \sum_i \int_{E_i} f .$$

Proof

Let $$u_i = f \cdot \chi_{E_i} .$$

Then $$f = f \cdot \chi_E = f \cdot \chi_{\bigcup E_i}$$

$$= f \cdot \sum_i \chi_{E_i} \text{ , since } E_i \text{ are disjoint}$$

$$= \sum_i f \cdot \chi_{E_i} \ ,$$

$$= \sum u_i ,$$

and so

$$\int_E f = \int_E \sum u_i$$

$$= \sum_i \int_E u_i \text{ , using Corollary (8.25)}$$

$$= \sum_i \int_E f \chi_{E_i}.$$

$$= \sum_i \int_{E_i} f.$$

(8.27) Definition A nonnegative measurable function f is called **integarble over the measurable set** E if

$$\int_E f < \infty.$$

(8.28) Proposition Let f and g be two nonnegative measurable functions. If f is integarble over E and $g(x) < f(x)$ on E, then g is also integarble on E, and

$$\int_E f - g = \int_E f - \int_E g.$$

Proof

By Proposition (8.22) in this section,

$$\int_E f = \int_E (f - g) + \int_E g. \qquad (1)$$

Since the left side is finite, the terms on the right must also be finite and so g is integarble. Also from (1) above, we have

$$\int_E f + g = \int_E f + \int_E g.$$

(8.29) Proposition Let f be a nonnegative function which is integarble over a set E. Then given $\varepsilon > 0$ there is a $\delta > 0$ such that for every set $A \subset E$ with $mA < \delta$ we have

$$\int_A f < \varepsilon.$$

Proof

The proposition would be trivial if f were bounded.

Set

$$f_n(x) = \begin{cases} f(x), \text{ if } f(x) \le n \\ n, \text{ otherwise} \end{cases}$$

Then each f_n is bounded (with a bound n). For each x as $n \to \infty$ we have $f_n(x) \to f(x)$. i.e., f_n converges to f at each point. By the Monotone Convergence Theorem there is an N such that $\int_E f_N > \int_E f - \frac{\varepsilon}{2}$, and $\int_E f - \int_E f_N < \frac{\varepsilon}{2}$. i.e., $\int_E f - f_N < \frac{\varepsilon}{2}$. Choose $\delta < \frac{\varepsilon}{2N}$. If $mA < \delta$, we have

$$\begin{aligned} \int_A f &= \int_A (f - f_N) + \int_A f_N \\ &< \int_E (f - f_N) + NmA \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

3 The General Lebesgue Integral

(8.30) Definition The **positive part** f^+ of a function f is the function

$$f^+ = f \vee 0;$$

i.e., f^+ is given by

$$f^+(x) = \max\{f(x), 0\}.$$

The **negative part** f^- of a function f is the function

$$f^- = (-f) \vee 0;$$

i.e., f^- is given by

$$f^-(x) = \max\{-f(x), 0\}.$$

(8.31) Remark If f is measurable, so are f^+ and f^-. We have

$$f = f^+ - f^-$$

and

$$|f| = f^+ + f^-.$$

(8.32) Definition A measurable function f is said to be **integrable** over E, if f^+ and f^- are both integrable over E. In this case we define

$$\int_E f = \int_E f^+ - \int_E f^-.$$

(8.33) Proposition Let f and g be integrable over E. Then

i. The function cf is integrable over E, and $\int_E cf = c\int_E f$.

ii. The function $f+g$ is integrable over E, and

$$\int_E f+g = \int_E f + \int_E g.$$

iii. If $f \le g$ a.e., then $\int_E f \le \int_E g$.

iv. If A and B are disjoint measurable sets contained in E, then

$$\int_{A\cup B} f = \int_A f + \int_B f.$$

Proof

Part (*i*) follows directly from the definition of the integral and Proposition (8.22). To prove part (*ii*), we first note that if f_1 and f_2 are nonnegative integrable functions with $f = f_1 - f_2$, then $f^+ + f_2 = f^- + f_1$. For Proposition (8.22) tells us that

$$\int f^+ + \int f_2 = \int f^- + \int f_1,$$

and so

$$\int f = \int f^+ - \int f^- = \int f_1 - \int f_2.$$

But, if f and g are integarble, so are $f^+ + g^+$ and $f^- + g^-$, and $f+g = \left(f^+ + g^+\right) - \left(f^- + g^-\right)$. Hence

$$\int (f+g) = \int \left(f^+ + g^+\right) - \int \left(f^- + g^-\right)$$

$$= \int f^+ + \int g^+ - \int f^- - \int g^-$$

$$= \int f + \int g.$$

Part (*iii*) follows from part (*ii*) and the fact that the integral of a nonnegative integrable function is nonnegative. For (*iv*) we have

$$\int_{A\cup B} f = \int f \chi_{A\cup B}$$

$$= \int f \chi_A + \int f \chi_B \text{ , as } A \text{ and } B \text{ are disjoint implies}$$

$$\chi_{A\cup B} = \chi_A + \chi_B$$

$$= \int_A f + \int_B f \ .$$

This completes the proof.

A and B are disjoint

(8.34) Lebesgue Convergence Theorem Let g be integrable over E and let $\langle f_n \rangle$ be a sequence of measurable functions such that $|f_n| \le g$ on E and for almost all x in E we have $f(x) = \lim f_n(x)$. Then

$$\int_E f = \lim \int_E f_n. \qquad \text{(UQ 2006)}$$

Proof

The function $g - f_n$ is nonnegative, and so by Fatou's Lemma

$$\int_E (g - f) \le \underline{\lim} \int_E (g - f_n).$$

Since $|f| \le g$, f is integrable, and we have

$$\int_E g - \int_E f \le \int_E g - \overline{\lim} \int_E f_n,$$

whence

$$\int_E f \ge \overline{\lim} \int_E f_n.$$

Similarly, considering $g + f_n$, we get

$$\int_E f \le \underline{\lim} \int_E f_n,$$

and the theorem follows.

(8.35) Theorem ***(Generalization of Lebesgue Convergence Theorem)*** Let $\langle g_n \rangle$ be a sequence of integrable functions which converges a.e. to an integrable function g. Let $\langle f_n \rangle$ be a sequence of measurable functions such that $|f_n| \le g_n$ and $\langle f_n \rangle$ converges to f a.e. If

$$\int g = \lim \int g_n,$$

then

$$\int f = \lim \int f_n.$$

Convergence in Measure

(8.36) Definition A sequence $\langle f_n \rangle$ of measurable functions is said to converge to f in measure if, given $\varepsilon > 0,$ there is an N such that for all $n \ge N$ we have

$$m\{x : |f(x) - f_n(x)| \ge \varepsilon\} < \varepsilon.$$

- It follows directly from this definition and Proposition (6.1) of Chapter 7 that if $\langle f_n \rangle$ is a sequence of measurable functions defined on a measurable set E of finite measure and $f_n \to f$ a.e., then $\langle f_n \rangle$ converges to f in measure.
- An example of a sequence $\langle f_n \rangle$ that converges to zero in measure in [0, 1] but such that $\langle f_n(x) \rangle$ does not converge for any x in [0, 1] can be constructed as follows: Let $n = k + 2^v,\ 0 \le k < 2^v,$ and set

$$f_n(x) = \begin{cases} 1, \text{ if } x \in [k2^{-v},\ (k+1)2^{-v}] \\ 0, \text{ otherwise} \end{cases}$$

Then $$m\{x : |f_n(x)| > \varepsilon\} \le \frac{2}{n},$$

and so $f_n \to 0$ in measure, although for any x in [0, 1], the sequence $\langle f_n(x) \rangle$ has the value 1 for arbitrarily large values of n and so does not converge.

(8.37) Proposition Let $\langle f_n \rangle$ be a sequence of measurable functions that converges in measure to f. Then there is a subsequence $\langle f_{n_k} \rangle$ that converges to almost everywhere.

(UQ 2006)

Proof

Given v, there is an integer n, such that for all $n \geq n_v$ we have

$$m\{x : |f_n(x) - f(x)| \geq 2^{-v}\} < 2^{-v}.$$

Let $E_v = \{x : |f_n(x) - f(x)| \geq 2^{-v}\}$. Then, if $x \notin \bigcup_{v=k}^{\infty} E_v$, we must have $|f_{n_v}(x) - f(x)| < 2^{-v}$ for $v \geq k$, and so $f_{n_v}(x) \to f(x)$. Hence $f_{n_v}(x) \to f(x)$ for any $x \notin A = \bigcap_{k=1}^{\infty} \bigcup_{v=k}^{\infty} E_v$. But $mA \leq m\left[\bigcup_{v=k}^{\infty} E_v\right] \leq \sum_{v=k}^{\infty} mE_v = 2^{-k+1}$. Hence $mA = 0$.

(8.38) Corollary Let $\langle f_n \rangle$ be a sequence of measurable functions defined on a measurable set E of finite measure. Then $\langle f_n \rangle$ converges to f in measure if and only if every subsequence of $\langle f_n \rangle$ has in turn a subsequence that converges almost everywhere to f.

(8.39) Proposition Fatou's Lemma and the Monotone and Lebesgue Convergence Theorems remain valid if 'convergence a.e.' is replaced by 'convergence in measure'.

Chapter 9

DIFFERENTIATION AND INTEGRATION

1 Differentiation of Monotone Functions

(1.1) Definition Let $\mathfrak{I}$ be a collection of intervals. We say that $\mathfrak{I}$ **covers** a set E in **the sense of Vitali**, if for each $\varepsilon > 0$ and any x in E, there is an interval $I \in \mathfrak{I}$ such that $x \in I$ and $l(I) < \varepsilon$. The intervals may be open, closed or half-open, but we do *not* allow degenerate intervals consisting of only one point.

(1.2) Lemma (Vitali) Let E be a set of finite outer measure and $\mathfrak{I}$ a collection of intervals that cover E in the sense of *VItali.* Then, given $\varepsilon > 0$, there is a finite disjoint collection $\{I_1,\ldots,I_N\}$ of intervals in $\mathfrak{I}$ such that

$$m^*\left[E \sim \bigcup_{n=1}^{N} I_n\right] < \varepsilon.$$

Proof

It suffices to prove the lemma in the case that each interval in $\mathfrak{I}$ is closed, for otherwise we replace each interval by its closure and observe that the set of endpoints of $I_1,\ldots,I_N$ has measure zero.

Let O be an open set of finite measure containing *E*. Since $\mathfrak{I}$ is a Vitali covering of *E*, we may assume without loss of generality that each $I \in \mathfrak{I}$ is contained in O. We choose a sequence $\langle I_n \rangle$ of disjoint intervals of $\mathfrak{I}$ by induction as follows: Let I_1 be any interval in $\mathfrak{I}$, and suppose that $I_1,\ldots,I_n$ has already been chosen. Let k_n be the supremum of the lengths of the intervals of $\mathfrak{I}$ that do not meet any of the intervals $I_1,\ldots,I_n$. Since each I is contained in O, we have $k_n \le mO < \infty$. Unless $E \subset \bigcup_{i=1}^{n} I_i$, we can find $I_{n+1} \in \mathfrak{I}$ with $l(I_{n+1}) > \frac{1}{2}k_n$ and I_{n+1} disjoint from $I_1,\ldots,I_n$.

Thus we have a sequence $\langle I_n \rangle$ of disjoint intervals of $\mathfrak{I}$, and since $\bigcup I_i \subset O$, we have $\sum l(I_n) \le mO < \infty$. Hence we can find an integer N such that

$$\sum_{N+1}^{\infty} l(I_n) < \frac{\varepsilon}{5}.$$

Let

$$R = E \sim \bigcup_{n=1}^{N} I_n.$$

The lemma will be established if we can show that $m^* R < \varepsilon$. Let x be an arbitrary point of R. Since $\bigcup_{n=1}^{N} I_n$ is a closed set not containing x, we can find an interval $I \in \Im$ which contains x and whose length is so small that I does not meet any of the intervals $I_1, \ldots, I_N$. If now $I \cap I_i = \Phi$ for $i \le n$, we must have $l(I) \le k_n < 2l(I_{n+1})$. Since $\lim l(I_n) = 0$, the interval I must meet at least one of the intervals I_n. Let n be the smallest integer such that I meets I_n. We have $n > N$, and $l(I) \le k_{n-1} \le 2l(I_n)$. Since $x \in I$, and I has a point in common with I_n, it follows that the distance from x to the midpoint of I_n is atmost $l(I) + \frac{1}{2} l(I_n) \le \frac{5}{2} l(I_n)$. Thus x belongs to the interval J_n having the same midpoint as I_n and five times the length. Thus we have shown that

$$R \subset \bigcup_{N+1}^{\infty} J_n.$$

Hence

$$m^* R \le \sum_{N+1}^{\infty} l(J_n) = 5 \sum_{N+1}^{\infty} l(J_n) < \varepsilon.$$

This completes the proof.

The four quantities - derivates

$$D^+ f(x) = \overline{\lim_{h \to 0^+}} \frac{f(x+h) - f(x)}{h},$$

$$D^- f(x) = \overline{\lim_{h \to 0^+}} \frac{f(x) - f(x-h)}{h},$$

$$D_+ f(x) = \underline{\lim_{h \to 0^+}} \frac{f(x+h) - f(x)}{h},$$

$$D_- f(x) = \underline{\lim_{h \to 0^+}} \frac{f(x) - f(x-h)}{h}.$$

Clearly, we have $D^+ f(x) \geq D_+ f(x)$ and $D^- f(x) \geq D_- f(x)$. If $D^+ f(x) = D_+ f(x) = D^- f(x) = D_- f(x) \neq \pm\infty$, we say that f **is differentiable at** x and define $f'(x)$ to be the common value of the derivates at x.

- If $D^+ f(x) = D_+ f(x)$, we say that f has a right-hand derivative at x and define $f'(x+)$ to be their common value.
- If $D^- f(x) = D_- f(x)$, we say that f has a left-hand derivative at x and define $f'(x-)$ to be their common value.

(1.3) Proposition If f is continuous on $[a,b]$ and one of its derivates (say D^+) is every where nonnegative on $[a,b]$, then f is nondecreasing on $[a,b]$; i.e., $f(x) \leq f(y)$ for $x \leq y$.

(1.4) Theorem Let f be an increasing real-valued function on the interval $[a,b]$.Then f is differentiable almost everywhere. The derivative f' is measurable, and

$$\int_a^b f'(x)dx \leq f(b) - f(a). \qquad \text{(UQ 2006)}$$

Proof

Let us show that the sets where any two derivates are unequal have measure zero. We consider only the set E where $D^+ f(x) > D_- f(x)$, the sets arising from other combinations of derivates being similarly handled. Now the set E is the union of the sets

$$E_{u,v} = \left\{x : D^+ f(x) > u > v > D_- f(x)\right\}$$

for all rational u and v. Hence it suffices to prove that $m^* E_{u,v} = 0$.

Let $s = m^* E_{u,v}$ and. choosing $\varepsilon > 0$, enclose $E_{u,v}$ in an open set O with $mO < s + \varepsilon$. For each point $x \in E_{u,v}$, there is an arbitrarily small interval $[x-h, x]$ contained in O such that

$$f(x) - f(x-h) < vh.$$

By Lemma (1.2) we can choose a finite collection $\{I_1, \ldots, I_N\}$ of them whose interiors cover a subset A of $E_{u,v}$ of outer measure greater than $s - \varepsilon$. Then, summing over these intervals, we have

$$\sum_{n=1}^{N} [f(x_n) - f(x_n - h_n)] < v \sum_{n=1}^{N} h_n$$

$$< vmO$$

$$< v(s+\varepsilon).$$

Now to each point $y \in A$ is the left endpoint of an arbitrarily small interval $(y,\ y+k)$ that is contained in some I_n and for which $f(y+k)-f(y)>uk.$ Using Lemma (1.2) again, we can pick out a finite collection $\{J_1,\ldots,J_M\}$ of such intervals such that their union contains a subset of A of outer measure greater than $s-2\varepsilon.$ Then summing over these intervals yields

$$\sum_{n=1}^{M}\left[f(y_i+k_i)-f(y_i)\right] > u\sum k_i$$

$$> u(s-2\varepsilon).$$

Each interval J_i is contained in some interval I_n, and if we sum over those i for which $J_i \subset I_n,$ we have

$$\sum\left[f(y_i+k_i)-f(y_i)\right] \le f(x_n)-f(x_n-h_n),$$

since f is increasing. Thus

$$\sum_{n=1}^{N} f(x_n)-f(x_n-h_n) \ge \sum_{i=1}^{M}\left[f(y_i+k_i)-f(y_i)\right],$$

and so

$$v(s+\varepsilon) > u(s-2\varepsilon).$$

Since this is true for each positive $\varepsilon,$ we have $vs \ge us.$ But $u>v,$ and so s must be zero.

This shows that

$$g(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

is defined almost everywhere and that f is differentiable whenever g is finite. Let

$$g_n(x)=n\left[f(x+\tfrac{1}{n})-f(x)\right],$$

where we set $f(x)=f(b)$ for $x \ge b.$ Then $g_n(x)\to g(x)$ for almost all $x,$ and so g is measurable. Since f is increasing, we have $g_n \ge 0.$ Hence by Fatou's Lemma

$$\int_a^b g \le \underline{\lim}\int_a^b g_n = \underline{\lim}\ n\int_a^b\left[f\left(x+\tfrac{1}{n}\right)-f(x)\right]dx$$

$$= \underline{\lim}\left[n\int_b^{b+\frac{1}{n}} f - n\int_a^{a+\frac{1}{n}} f\right]$$

$$= \underline{\lim} \left[f(b) - n \int_a^{a+\frac{1}{n}} f \right]$$

$$\leq f(b) - f(a).$$

This shows that g is integarble and hence finite almost everywhere. Thus f is differentiable a.e. and $g = f'$ a.e.

2 Functions of Bounded Variation

Let us denote

$$r^+ = \begin{cases} r, \text{ if } r \geq 0 \\ 0, \text{ if } r \leq 0 \end{cases}$$

$$r^- = |r| - r^+.$$

Then r^+ and r^- are non-negative and $r = r^+ - r^-$.

For example, if $r = 5$, then $r^+ = 5$ and $r^- = |r| - r^+ = 5 - 5 = 0$; if $r = -5$, then $r^+ = 0$ and $r^- = |r| - r^+ = 5 - 0 = 5$.

Let f be a real-valued function defined on the interval $[a, b]$, and let $a = x_0 < x_1 < \cdots < x_k = b$ be any subdivision of $[a, b]$.

Define

$$p = \sum_{i=1}^{k} \left[f(x_i) - f(x_{i-1}) \right]^+$$

$$n = \sum_{i=1}^{k} \left[f(x_i) - f(x_{i-1}) \right]^-$$

$$t = n + p = \sum_{i=1}^{k} \left| f(x_i) - f(x_{i-1}) \right|.$$

We have $$f(b) - f(a) = p - n,$$

since $p - n = \sum_{i=1}^{k} \left[f(x_i) - f(x_{i-1}) \right]^+ - \sum_{i=1}^{k} \left[f(x_i) - f(x_{i-1}) \right]^- = \sum_{i=1}^{k} \left[f(x_i) - f(x_{i-1}) \right] = f(x_k) - f(x_0).$

Set

$$P = \sup p,$$

$$N = \sup n,$$

$$T = \sup t,$$

where we take the suprema over all possible subdivisions of $[a, b]$. We clearly have $P \le T \le P + N$. We call P, N, T the positive, negative, and total variations of f over $[a, b]$. We some times write T_a^b, $T_a^b(f)$, etc., to denote the dependence on the interval $[a, b]$ or on the function f. If $T < \infty$, we say that f is of **bounded variation** over $[a, b]$. This notion is abbreviated by writing $f \in BV$.

(2.1) Notation P_a^b denotes $\sup p$ where we take the suprema over all possible subdivisions of $[a, b]$. P_a^x denotes $\sup p$ where we take the suprema over all possible subdivisions of $[a, x]$. Similarly, N_a^b, N_a^x etc denote suprema over all possible subdivisions of $[a, b]$ or $[a, x]$.

(2.2) Lemma If f is of bounded variation on $[a, b]$, then

$$T_a^b = P_a^b + N_a^b$$

and

$$f(b) - f(a) = P_a^b - N_a^b.$$

Proof

For any subdivision of $[a, b]$

$$p = n + f(b) - f(a),$$

and taking suprema over all possible subdivisions, we obtain

$$P = N + f(b) - f(a).$$

Also

$$t = p + n = p + p - \{f(b) - f(a)\}.$$

Taking surpema, we obtain

$$T = 2P - \{f(b) - f(a)\} = P + N.$$

(2.3) Theorem A function f is of bounded variation on $[a, b]$ if and only if f is the difference of two monotone real-valued functions on $[a, b]$.

Proof

Let f be of bounded variation, and set $g(x) = P_a^x$ and $h(x) = N_a^x$. Then g and h are monotone increasing functions which are real valued, since $0 \le P_a^x \le T_a^x \le T_a^b < \infty$ and

$0 \le N_a^x \le T_a^x \le T_a^b < \infty$. But $f(x) = g(x) - h(x) + f(a)$ by Lemma (2.2). Since $h - f(a)$ is a monotone function, we have f expressed as the difference of two monotone functions.

On the other hand, if $f = g - h$ on $[a, b]$ with g and h increasing, then for any subdivision we have

$$\sum |f(x_i) - f(x_{i-1})| \le \sum |g(x_i) - g(x_{i-1})| + \sum |h(x_i) - h(x_{i-1})|$$
$$= g(b) - g(a) + h(b) - h(a).$$

Hence

$$T_a^b(f) \le g(b) + h(b) - g(a) - h(a).$$

(2.4) Corollary If f is of bounded variation on $[a, b]$, then $f'(x)$ exists for almost all $x \in [a, b]$.

3 Differentiation of an Integral

(3.1) Definition If f is an integrable function on $[a, b]$, we define its **indefinite integral** to be the function F defined on $[a, b]$ by

$$F(x) = \int_a^x f(t)\,dt.$$

(3.2) Lemma If f is an integrable function on $[a, b]$, then the function F defined by

$$F(x) = \int_a^x f(t)\,dt$$

is a continuous function of bounded variation on $[a, b]$. (UQ 2006)

Proof

Clearly F is continuous (Details are left to the exercise). To show that F is of bounded variation, let $a = x_0 < x_1 < \cdots < x_k = b$ be any subdivision of $[a, b]$. Then

$$\sum_{i=1}^{k} |F(x_i) - F(x_{i-1})| = \sum_{i=1}^{k} \left| \int_{x_{i-1}}^{x_i} f(t)\,dt \right| \le \sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} |f(t)|\,dt$$

$$= \int_a^b |f(t)|\,dt.$$

Thus $$T_a^b(F) \le \int_a^b |f(t)|\,dt < \infty.$$

(3.3) Lemma If f is integrable on $[a, b]$, and

$$\int_a^x f(t)\,dt = 0$$

for all $x \in [a, b]$, then $f(t) = 0$ a.e. in $[a, b]$.

Proof

Suppose $f(x) > 0$ on a set E of positive measure. Then, by Proposition (4.4) in Chapter 7 *Lebesgue Meausre,* there is a closed set $F \subset E$ with $mF > 0$. Let $O = (a,b) \setminus F$. Then either $\int_a^b f \neq 0,$ or else

$$0 = \int_a^b f = \int_F f + \int_O f,$$

and

$$\int_O f = -\int_F f \neq 0.$$

But O is the disjoint union of a countable collection $\{(a_n, b_n)\}$ of open intervals, and so by Proposition: **"**Let f be a nonnegative function and $\langle E_i \rangle$ a disjoint sequence of measurable sets. Let $E = \bigcup E_i$. Then $\int_E f = \sum_i \int_{E_i} f$ **"** , we have

$$\int_O f = \sum \int_{a_n}^{b_n} f.$$

Thus for some n we have

$$\int_{a_n}^{b_n} f \neq 0,$$

and so either

$$\int_a^{a_n} f \neq 0,$$

or

$$\int_a^{b_n} f \neq 0.$$

In any case we see that if f is positive on a set of positive measure, then for some $x \in [a, b]$ we have

$$\int_a^x f \neq 0.$$

Similarly, if f negative on a set of positive measure, i.e., $f(x) < 0$ on a set E of positive measure, then for some $x \in [a, b]$ we have

$$\int_a^x f \neq 0.$$

i.e., in any case $f(x) \neq 0$ on a set of positive measure implies $\int_a^x f \neq 0.$

Hence by contraposition[†], it follows that $\int_a^x f(t)\,dt = 0$ for all $x \in [a, b]$, then $f(t) = 0$ a.e. in $[a, b]$.

[†] Contraposition: If p and q are two statements, then $p \Rightarrow q$ and $\sim q \Rightarrow \sim p$ are logically equivalent. Hence to show that $p \Rightarrow q$ it is enough to show that $\sim q \Rightarrow \sim p$. In this Lemma

the atement $p : \int_a^x f(t)\,dt = 0$ for all $x \in [a, b]$,

statement $q :$ $f(t) = 0$ a.e. in $[a, b]$.

statement $\sim q$: $f(t) \neq 0$ on a set of positive measure

statement $\sim p$: $\int_a^x f(t)\,dt \neq 0$ for some $x \in [a, b]$.

(3.4) Lemma If f is bounded and measurable on $[a, b]$ and

$$F(x) = \int_a^x f(t)\,dt + F(a),$$

then $F'(x) = f(x)$ for almost all x in $[a, b]$. (UQ 2006)

Proof

By Lemma (3.2), F is bounded variation over $[a, b]$, and so $F'(x)$ exists for almost all x in $[a, b]$. Let $|f| \leq K$. Then setting

$$f_n(x) = \frac{F(x+h) - F(x)}{h},$$

with $h = \frac{1}{n}$, we have

$$f_n(x) = \frac{1}{h}\int_x^{x+h} f(t)dt,$$

and so $$|f_n| \le K.$$

Since $f_n(x) \to F'(x)$ a.e., the bounded convergence theorem implies that

$$\int_a^c F'(x)dx = \lim \int_a^c f_n(x)dx = \lim_{h\to 0} \frac{1}{h}\int_a^c \big[F(x+h) - F(x)\big]dx$$

$$= \lim_{h\to 0}\left[\frac{1}{h}\int_a^{c+h} F(x)dx - \frac{1}{h}\int_a^{a+h} F(x)dx\right]$$

$$= F(c) - F(a) = \int_a^c f(x)dx,$$

since F is continuous. Hence

$$\int_a^c \{F'(x) - f(x)\}\,dx = 0$$

for all $c \in [a, b]$, and so

$$F'(x) = f(x) \qquad \text{a.e. by Lemma (3.3).}$$

This completes the proof.

(3.5) Theorem Let f be an integrable function on $[a, b]$, and suppose that

$$F(x) = F(a) + \int_a^x f(t)\,dt.$$

Then $F'(x) = f(x)$ for almost all x in $[a, b]$.

Proof

Without loss of generality, we may assume $f \ge 0$. Let f_n be defined by $f_n(x) = f(x)$, if $f(x) \le n$, and $f_n(x) = n$ if $f(x) > n$. Then $f - f_n \ge 0$, and so

$$G_n(x) = \int_a^x f - f_n$$

is an increasing function of x, which must have a derivative almost everywhere, and this derivative will be nonnegative. Now by Lemma (3.4)

$$\frac{d}{dx}\int_a^x f_n = f_n(x) \text{ a.e.},$$

and so

$$F'(x) = \frac{d}{dx}G_n + \int_a^x f_n$$

$$\geq f_n(x) \quad \text{a.e.}$$

Since n is arbitrary,

$$F'(x) \geq f(x) \quad \text{a.e.}$$

Consequently,

$$\int_a^b F'(x)dx \geq \int_a^b f(x)dx = F(b) - F(a).$$

Thus by Theorem (1.4) we have

$$\int_a^b F'(x)dx = F(b) - F(a) = \int_a^b f(x)dx$$

and

$$\int_a^b \{F'(x) - f(x)\}\, dx = 0.$$

Since $F'(x) - f(x) \geq 0,$ this implies that $F'(x) - f(x) = 0$ a.e., and so $F'(x) = f(x)$ a.e.

4 Absolute Continuity

(4.1) Definition A real-valued function f defined on $[a, b]$ is said to be **absolutely continuous** on $[a, b]$ if, given $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\sum_{i=1}^{n} \left| f(x_i') - f(x_i) \right| < \varepsilon$$

for every finite collection $\left\{(x_i, x_i')\right\}$ of nonoverlapping intervals with

$$\sum_{i=1}^{n} \left| x_i' - x_i \right| < \delta.$$

(4.2) Remarks

- An absolutely continuous function is continuous.
- Every indefinite integral is absolutely continuous, by the Proposition : "Let f be a nonnegative function which is integrable over a set E. Then given $\varepsilon > 0$ there is a $\delta > 0$ such that for every set $A \subset E$ with $mA < \delta$ we have $\int_A f < \varepsilon$."

(4.3) Lemma If f is absolutely continuous on $[a, b]$, then it is of bounded variation on $[a, b]$.

Proof

Let δ be the δ in the definition of absolute continuity that corresponds to $\varepsilon = 1$. Then any subdivision of $[a, b]$ can be split (by inserting fresh division points, if necessary) into K sets of intervals, each of total length less than δ, where K is the largest integer less than $1 + \frac{(b-a)}{\delta}$. Thus for any subdivision we have $t \le K$, and so $T \le K$.

(4.4) Corollary If f is absolutely continuous, then f has a derivative almost every where.

Proof

By Lemma (4.3), f is absolutely continuous implies f is of bounded variation and then by Corollary (2.4), f has a derivative almost every where.

(4.5) Lemma If f is absolutely continuous on $[a, b]$ and $f'(x) = 0$ a.e., then f is constant.

Proof

We wish to show that $f(a) = f(c)$ for any $c \in [a, b]$. Let $E \subset (a, c)$ be set of measure $c - a$ in which $f'(x) = 0$, and let ε and η be arbitrary positive numbers. To each $x \in E$ there is an arbitrarily small interval $[x, x+h]$ contained in $[a, c]$ such that $|f(x+h) - f(x)| < \eta h$. By Lemma (1.2) we can find a finite collection $\{[x_k, y_k]\}$ of nonoverlapping intervals of this sort which cover all of E except for a set of measure less than δ, where δ is the positive number corresponding to ε in the definition of the absolute continuity of f. If we label the x_k so that $x_k \le x_{k+1}$, we have

$$y_0 = a \le x_1 < y_1 \le x_2 < \cdots \le y_n \le c = x_{n+1}$$

and

$$\sum_{k=0}^{n} |x_{k+1} - y_k| < \delta.$$

Now

$$\sum_{k=0}^{n} |f(y_k) - f(x_k)| \le \eta \sum (y_k - x_k)$$

$$< \eta(c-a)$$

by the way the intervals $\{[x_k, y_k]\}$ were constructed, and

$$\sum_{k=0}^{n} |f(x_{k+1}) - f(y_k)| < \varepsilon$$

by the absolute continuity of f. Thus

$$|f(c) - f(a)| = \sum_{k=0}^{n} [f(x_{k+1}) - f(y_k)] + \sum_{k=1}^{n} [f(y_k) - f(x_k)]$$

$$\le \varepsilon + \eta(c-a).$$

Since ε and η are arbitrary positive numbers, $f(c) - f(a) = 0$.

(4.6) Theorem A function F is an indefinite integral if and only if it is absolutely continuous.

Proof

If F is an indefinite integral, then F is absolutely continuous by the Proposition : "Let f be a nonnegative function which is integrable over a set E. Then given $\varepsilon > 0$ there is a $\delta > 0$ such that for every set $A \subset E$ with $mA < \delta$ we have $\int_A f < \varepsilon$."

Suppose on the other hand that F is absolutely continuous on $[a, b]$. Then F is of bounded variation, and we may write

$$F(x) = F_1(x) - F_2(x),$$

where the functions F_i are monotone increasing. Hence $F'(x)$ exists almost everywhere and

$$|F'(x)| \le F_1'(x) + F_2'(x).$$

Thus

$$\int |F'(x)|\, dx \le F_1(b) + F_2(b) - F_1(a) - F_2(a)$$

by Theorem (1.4), and $F'(x)$ is integrable. Let

$$G(x) = \int_a^x F'(t)\, dt.$$

Then G is absolutely continuous and so is the function $f = F - G.$ It follows from Theorem (3.5) that $f'(x) = F'(x) - G'(x) = 0$ a.e., and so f is constant by Lemma (4.5). Thus

$$F(x) = \int_a^x F'(t)dt + F(a). \tag{1}$$

i.e., F is an indefinite integral.

(4.7) Corollary Every absolutely continuous function is the indefinite integral of its derivative.

Proof

Proof is obvious from (1) of Theorem (4.6), for if F is absolutely continuous function then $F(x) = \int_a^x F'(t)dt + F(a),$ which says that F is the indefinite integral of its derivative F'.

5 Convex Functions

(5.1) Definition A function φ defined on an open interval (a, b) is said to be **convex** if for each $x, y \in (a, b)$ and each λ, $0 \le \lambda \le 1$ we have

$$\varphi\big(\lambda x + (1-\lambda)y\big) \le \lambda\varphi(x) + (1-\lambda)\varphi(y).$$

If we look at the graph of φ in $\mathbb{R}^2$, this condition can be formulated geometrically by saying that each point on the chord between $\langle x, \varphi(x)\rangle$ and $\langle y, \varphi(y)\rangle$ is above the graph of φ.

(5.2) Lemma If φ is convex on (a,b) and if x, y, x', y' are points of (a,b) with $x \le x' < y'$ and $x < y \le y'$, then the chord over (x', y') has larger slope than the chord over (x, y); i.e.,

$$\frac{\varphi(y) - \varphi(x)}{y - x} \le \frac{\varphi(y') - \varphi(x')}{y' - x'}.$$

(5.3) Definition If the upper and lower left-hand derivatives $D^- f$ and $D_- f$ of a function f are equal and finite at a point x, we say that ***f* is differentiable on the left at *x*** and call this common value the left-hand derivative at x. Similarly, we say that f is differentiable on the right at x if $D^+ f$ and $D_+ f$ are equal there.

(5.4) Proposition If φ is convex on (a,b), then φ is absolutely continuous on each closed subinterval of (a,b). The right- and left-hand derivatives of φ exist at each point of (a,b) and are equal to each other except on a countable set. The left- and right-hand

derivatives are monotone increasing functions, and at each point the left-hand derivative is less than or equal to the right-hand derivative.

Proof

Let $[c, d] \subset (a, b)$. Then, by Lemma (5.2), we have

$$\frac{\varphi(c)-\varphi(a)}{c-a} \le \frac{\varphi(y)-\varphi(x)}{y-x} \le \frac{\varphi(b)-\varphi(d)}{b-d}$$

for $x, y \in [c, d]$. Thus $|\varphi(y)-\varphi(x)| \le M|x-y|$ in $[c, d]$, and so φ is absolutely continuous there.

If $x_0 \in (a, b)$, then $\dfrac{\varphi(x)-\varphi(x_0)}{x-x_0}$ is an increasing function of x by Lemma (5.2), and so the limits as x approaches x_0 from the right and from the left exist and are finite. Thus φ is differentiable on the right and on the left at each point, and the left-hand derivative is less than or equal to the right-hand derivative. If $x_0 < y_0$, $x < y_0$ and $x_0 < y$, then

$$\frac{\varphi(x)-\varphi(x_0)}{x-x_0} \le \frac{\varphi(y)-\varphi(y_0)}{y-y_0},$$

and either derivative at x_0 is less than or equal to either derivative at y_0. Consequently, each derivative is monotone, and they are equal at a point if one of them is continuous there. Since a monotone function can have only a countable number of discontinuities, they are equal except on a countable set.

(5.5) Proposition If φ is a continuous function on (a, b) and if one derivative (say D^+) of φ is nondecreasing, then φ is convex.

Proof

Given x, y with $a < x < y < b$, define a function ψ on $[0, 1]$ by

$$\psi(t) = \varphi[ty + (1-t)x] - t\varphi(y) - (1-t)\varphi(x).$$

Our goal is to show that ψ is nonpositive on $[0, 1]$. Now ψ is continuous, and $\psi(0) = \psi(1) = 0$. Moreover,

$$D^+\psi = (y-x)D^+\varphi - \varphi(y) + \varphi(x),$$

and so $D^+\psi$ is nondecreasing on $[0, 1]$.

Let γ be a point where ψ assumes its maximum on $[0, 1]$. If $\gamma = 1$, then $\psi(t) \le \psi(1) = 0$ on $[0, 1]$. Hence suppose that $\gamma \in [0, 1)$.

Since ψ has a local maximum at γ, we have $D^+\psi(\gamma) \le 0$. But $D^+\psi$ was nondecreasing, and so $D^+\psi \le 0$ on $[0, \gamma]$. Consequently, ψ is nonincreasing on $[0, \gamma]$, and hence $\psi(\gamma) \le \psi(0) = 0$. Thus the maximum of ψ on $[0, 1]$ is nonpositive, and so $\psi \le 0$ on $[0, 1]$.

(5.6) Proposition Let φ have a second derivative at each point of (a, b). Then φ is convex on (a, b) if and only if $\varphi''(x) \ge 0$ for each $x \in (a, b)$.

Let φ be a convex function on (a, b) and $x_0 \in (a, b)$. The line $y = m(x - x_0) + \varphi(x_0)$ through $\langle x_0, \varphi(x_0)\rangle$ is called a **supporting line** at x_0 if it always lies below the graph of φ, that is, if

$$\varphi(x) \ge m(x - x_0) + \varphi(x_0).$$

It follows from Lemma (5.2) that such a line is a supporting line if and only if its slope m lies between the left- and right-hand derivatives at x_0. Thus, in particular, there is always at least one supporting line at each point. This notion enables us to give a short proof for the following proposition:

(5.7) Proposition (Jensen Inequality) Let φ be a convex function on $(-\infty, \infty)$ and f an integrable function on $[0, 1]$. Then

$$\int \varphi(f(t))dt \ge \varphi\left[\int f(t)\,dt\right].$$

Proof

Let $\alpha = \int f(t)\,dt$, and let $y = m(x - \alpha) + \varphi(\alpha)$ be the equation of a supporting line at α. Then

$$\varphi(f(t)) \ge m\big(f(t) - \alpha\big) + \varphi(\alpha).$$

Integrating both sides with respect to t gives the proposition.

(5.8) Corollary Let f be an integrable function on $[0, 1]$. Then

$$\int \exp(f(t))dt \ge \exp\left[\int f(t)\,dt\right].$$

(5.9) Definition A function φ defined on an open interval (a, b) is said to be **strictly convex** if for each $x, y \in (a, b)$ and each λ, $0 < \lambda < 1$ we have

$$\varphi\big(\lambda x + (1 - \lambda)y\big) < \lambda\varphi(x) + (1 - \lambda)\varphi(y).$$

(5.10) Definition A function φ defined on an open interval (a, b) is said to be **concave** if $-\varphi$ is convex.

(5.11) Remark The only functions that are both convex and concave are the linear functions.

(5.12) Definition If I is any interval, (open, closed, or half-open), we say that φ is convex on I if φ is continuous on I and convex in the interior.

Chapter 10

MEASURE AND INTEGRATION

1 Measure Spaces

The purpose of the present chapter is to abstract the most important properties of Lebesgue measure and Lebesgue integration. We shall do this by giving certain axioms which Lebesgue measure satisfies and base our integration theory on these axioms. As a consequence our theory will be valid for every system satisfying the given axioms.

(1.1) Definition A **σ-algebra** $\mathcal{B}$ is a family of subsets of a given set X which contains Φ and is closed with respect to complements and with respect to countable unions.

(1.2) Definition By a **set function** μ we mean a function which assigns an extended real number to certain sets.

(1.3) Definition By a **measurable space** we mean a couple $(X, \mathcal{B})$ consisting of a set X and $\mathcal{B}$ of subsets of X. A subset A of X is called measurable (or measurable with respect to $\mathcal{B}$) if $A \in \mathcal{B}$.

(1.4) Definition By a **measure** μ on a measurable space $(X, \mathcal{B})$ we mean a nonnegative set function defined for all sets of $\mathcal{B}$ and satisfying $\mu(\Phi) = 0$ and

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu E_i \tag{1}$$

for any sequence E_i of disjoint measurable sets. By a measure space $(X, \mathcal{B}, \mu)$ we mean a measurable space $(X, \mathcal{B})$ together with a measure μ defined on $\mathcal{B}$.

The property (1) of μ is often referred to by saying that μ is **countably additive.** We also have that μ is **finitely additive**; i.e.,

$$\mu\left(\bigcup_{i=1}^{N} E_i\right) = \sum_{i=1}^{N} \mu E_i, \tag{2}$$

for disjoint sets E_i belong to $\mathcal{B}$, since we may set $E_i = \Phi$ for $i > N$.

(1.5) Examples

1. $(\mathbb{R}, \mathcal{M}, m)$ is a measure space, where $\mathbb{R}$ is the set of real numbers, $\mathcal{M}$ the Lebesgue measurable sets of real numbers, and m Lebesgue measure.
2. $([0, 1], \mathcal{M}, m)$ is a measure space, where $\mathcal{M}$ is the measurable subsets of sets of $[0, 1]$ real numbers, and m Lebesgue measure.
3. $(\mathbb{R}, \mathcal{B}, m)$ is a measure space, where $\mathcal{B}$ is the class of Borel sets, and m Lebesgue measure.
4. Let X be any uncountable set, $\mathcal{B}$ the family of those subsets which are either countable or the complement of a countable set. Then $\mathcal{B}$ is a σ-algebra and we can define a measure on it by setting $\mu A = 0$ for each countable set and $\mu B = 1$ for each set whose complement is countable.

(1.6) Proposition If $A \in \mathcal{B}$, $B \in \mathcal{B}$ and $A \subset B$, then

$$\mu A \leq \mu B.$$

Proof

Since $A \cap [B \setminus A] = \Phi$,

$$B = A \cup [B \setminus A]$$

is a disjoint union, and hence we have

$$\mu B = \mu A + \mu(B \setminus A) \geq \mu A, \text{ since } \mu(B \setminus A) \geq 0.$$

(1.7) Proposition If $E_i \in \mathcal{B}$, $\mu E_1 < \infty$ and $E_i \supset E_{i+1}$, then

$$\mu\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n \to \infty} \mu E_n. \tag{3}$$

Proof

Set $E = \bigcap_{i=1}^{\infty} E_i$. Then

$$E_1 = E \cup \bigcup_{i=1}^{\infty} (E_i \setminus E_{i+1})$$

and this is a disjoint union. Hence

$$\mu E_1 = \mu E + \sum_{i=1}^{\infty} \mu (E_i \setminus E_{i+1}).$$

Since

$$E_i = E_{i+1} \cup (E_i \setminus E_{i+1})$$

is a disjoint union, we have

$$\mu\left(E_i \setminus E_{i+1}\right) = \mu E_i - \mu E_{i+1}.$$

Hence

$$\begin{aligned}\mu E_1 &= \mu E + \sum_{i=1}^{\infty}\left(\mu E_i - \mu E_{i+1}\right)\\ &= \mu E + \lim_{n\to\infty}\sum_{i=1}^{n-1}\left(\mu E_i - \mu E_{i+1}\right)\\ &= \mu E + \mu E_1 - \lim_{n\to\infty}\mu E_n,\end{aligned}$$

whence the proposition follows.

(1.8) Proposition If $E_i \in \mathcal{B}$, then

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \le \sum_{i=1}^{\infty}\mu E_i.$$

Proof

Let

$$G_n = E_n \setminus \left[\bigcup_{i=1}^{n-1} E_i\right].$$

Then $G_n \subset E_n$ and the sets G_n are disjoint. Hence

$$\mu G_n \le \mu E_n,$$

while

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \le \sum_{i=1}^{\infty}\mu G_n \le \sum_{n=1}^{\infty}\mu E_n.$$

(1.9) Definition A measure μ is called **finite** if $\mu(X) < \infty$. It is called a σ**-finite** if there is a sequence $\langle X_n \rangle$ of measurable sets in $\mathcal{B}$ such that

$$X = \bigcup_{n=1}^{\infty} X_n$$

and $\mu X_n < \infty$.

(1.10) Examples

1. Lebesgue measure on [0, 1] is of finite measure.
2. Lebesgue measure on $(-\infty, \infty)$ is a σ - finite measure.
3. The counting measure on an uncountable set is a measure that is <u>not</u> σ-finite.

(1.11) Definition A set E is said to be of **finite measure** if $E \in \mathcal{B}$ and $\mu E < \infty$. A set E is said to be of σ-**finite measure** if E is the union of a countable collection of measurable sets of finite measure.

Any measurable set contained in a set of σ-finite measure is itself and the union of a countable collection of sets of σ-finite measure is again of σ-finite measure. If μ is σ-finite, then every measurable set is of σ-finite measure.

(1.12) Definition A measure μ is said to be **semifinite** if each measurable set of infinite measure contains measurable sets of arbitrary large finite measure.

- Every σ-finite measure is semifinite.
- The measure that assigns 0 to countable subsets of an uncountable set X and ∞ to the uncountable sets is <u>not</u> semifinite.

(1.13) Definition A measure space $(X, \mathcal{B}, \mu)$ is said to be **complete** if $\mathcal{B}$ contains all subsets of sets of measure zero, that is, if $B \in \mathcal{B}$, $\mu B = 0$, and $A \subset B$ imply $A \in \mathcal{B}$.

- Lebesgue measure is complete.
- Lebesgue measure restricted to the σ-algebra of Borel sets is <u>not</u> complete.

(1.14) Proposition If $(X, \mathcal{B}, \mu)$ is a measure space, then we can find a complete measure space $(X, \mathcal{B}_0, \mu_0)$ such that

(i) $\mathcal{B} \subset \mathcal{B}_0$.

(ii) $E \in \mathcal{B} \Rightarrow \mu E = \mu_0 E$.

(iii) $E \in \mathcal{B}_0 \Leftrightarrow E = A \cup B$ where $B \in \mathcal{B}$ and $A \subset C$, $C \in \mathcal{B}$, $\mu C = 0$.

The measure space $(X, \mathcal{B}_0, \mu_0)$ is called the **completion** of $(X, \mathcal{B}, \mu)$.

(1.15) Definition If $(X, \mathcal{B}, \mu)$ is a measure space, we say that a subset E of X is locally measurable if $E \cap B \in \mathcal{B}$ for each $B \in \mathcal{B}$ with $\mu B < \infty$. The collection $\mathcal{C}$ of all locally measurable sets is a σ-algebra containing $\mathcal{B}$. The measure μ is called **saturated** if every locally measurable set is measurable (i.e., is in $\mathcal{B}$).

- Every σ-finite measure is saturated.
- A measure can always be extended to a saturated measure, but unlike the process of completion, the process of saturation is not uniquely determined.

2 Measurable Functions

The concept of a measurable function on an abstract measurable space is almost identical with that for functions of a real variable. Consequently, those propsoitons and theorem whose proofs are essentially the same as those in Chapter 7 *Lebesgue Measure* are stated without proof:

Throughout the section we assume that a fixed measurable space $(X, \mathcal{B})$ is given

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(2.1) Proposition ***(Analogue of Proposition (5.1) in Chapter 7)*** Let f be an extended real-valued function defined on X (where $(X, \mathcal{B})$ is a measurable space). Then the following statements are equivalent:

i. For each real number α the set $\{x : f(x) > \alpha\} \in \mathcal{B}$.

ii. For each real number α the set $\{x : f(x) \geq \alpha\} \in \mathcal{B}$.

iii. For each real number α the set $\{x : f(x) < \alpha\} \in \mathcal{B}$.

iv. For each real number α the set $\{x : f(x) \leq \alpha\} \in \mathcal{B}$.

(2.2) Definition The extended real-valued function f defined on X is called **measurable** (or **measurable with respect to** $\mathcal{B}$) if any one of the statements of Proposition (2.1) holds.

Recall that $f \vee g$ is defined by

$$(f \vee g)(x) = \max\{f(x),\ g(x)\}.$$

(2.3) Theorem ***(Analogue of Proposition (6.7) and Theorem (6.12) in Chapter 7)*** If c is a constant and the functions f and g are measurable, then so are the functions $f + c,\ cf,\ f + g,\ f \cdot g,$ and $f \vee g.$ Moreover, if $\langle f_n \rangle$ is a sequence of measurable functions, then $\sup\limits_n f_n,\ \inf\limits_n f_n,\ \overline{\lim} f_n$, and $\underline{\lim} f_n$ are all measurable.

Recall that a **simple function** is a finite linear combination

$$\varphi(x) = \sum_{i=1}^{n} c_i \chi_{E_i}(x)$$

of characteristic functions of measurable sets E_i.

(2.4) Proposition Let f be a nonnegative measurable function. Then there is a sequence $\langle \varphi_n \rangle$ of simple functions with $\varphi_{n+1} \geq \varphi_n$ such that $f = \lim \varphi_n$ at each point of X. If f is defined on a σ-finite measure space, then we may choose the functions φ_n so that each vanishes outside a set of finite measure.

(2.5) Proposition ***(Analogue of Proposition (6.14) in Chapter 7)*** If μ is a complete measure and f is a measurable function, then $f = g$ a.e. implies g is measurable.

The sets $\{x : f(x) < \alpha\}$ are sometimes called ordinate sets for f. They increase with α. The following lemma states that, given a collection $\{B_\alpha\}$ of measurable sets that increase with α, we can find a measurable function f which nearly has these for ordinate sets in the sense that

$$\{x : f(x) < \alpha\} \subset B_\alpha \subset \{x : f(x) \leq \alpha\}.$$

(2.6) Lemma Suppose that to each α in a dense set D of real numbers there is assigned a set $B_\alpha \in \mathcal{B}$ such that $B_\alpha \subset B_\beta$ for $\alpha < \beta$. Then there is a unique measurable extended real-valued function f on X such that $f \leq \alpha$ on B_α and $f \geq \alpha$ on $X \setminus B_\alpha$.

Proof

For each $x \in X$ define

$$f(x) = \inf \{\alpha \in D : x \in B_\alpha\}$$

where, as usual, $\inf \Phi = \infty$. If $x \in B_\alpha$, then $f(x) \leq \alpha$. If $x \notin B_\alpha$, then $x \notin B_\beta$ for each $\beta < \alpha$, and so $f(x) \geq \alpha$. To show that f is measurable, we take $\lambda \in$ □ and choose a sequence $\langle \alpha_n \rangle$ from D with $\alpha_n < \lambda$ and $\lambda = \lim \alpha_n$. Then

$$\{x : f(x) < \lambda\} = \bigcup_{n=1}^{\infty} B_{\alpha_n}.$$

For if $f(x) < \lambda$, then $f(x) < \alpha_n$ for some n, and so $x \in B_{\alpha_n}$ for some n, then $f(x) \leq \alpha_n < \lambda$. Thus the sets $\{x : f(x) < \lambda\}$ are all measurable, and so f is measurable.

To prove the uniqueness of f, let g be any extended real-valued function with $g \leq \alpha$ on B_α and $g \geq \alpha$ on $\tilde{B}_\alpha$. Then $x \in B_\alpha$ implies $g(x) \leq \alpha$, and so

$$\{\alpha \in D : x \in B_\alpha\} \subset \{\alpha \in D : \alpha \geq g(x)\}.$$

Since $g(x) < \alpha$ implies that $x \in B_\alpha$ we have

$$\{\alpha \in D : \alpha > g(x)\} \subset \{\alpha \in D : x \in B_\alpha\}.$$

Because of the density of D we have

$$\begin{aligned} g(x) &= \inf\{\alpha \in D : \alpha > g(x)\} \\ &= \inf\{\alpha \in D : \alpha \geq g(x)\} \\ &= \inf\{\alpha \in D : x \in B_\alpha\} = f(x). \end{aligned}$$

(2.7) Proposition Suppose that for each α in a dense set D of real numbers there is assigned a set $B_\alpha \in \mathcal{B}$ such that $\mu(B_\alpha \setminus B_\beta) = 0$ for $\alpha < \beta$. Then there is a measurable function f such that $f \leq \alpha$ a.e. on B_α and $f \geq \alpha$ a.e. on $X \setminus B_\alpha$. If g is any other function with this property, then $g = f$ a.e.

Proof

Let C be a countable dense subset of D, and set $N = \bigcup(B_\alpha \setminus B_\beta)$ for α and β in C wth $\alpha < \beta$. Then N is the countable union of sets of measure zero and so itself a set of measure zero. Let $B'_\alpha = B_\alpha \cup N$.. For α and β in C with $\alpha < \beta$ we have

$$B'_\alpha \setminus B'_\beta = (B_\alpha \cup N) \setminus (B_\beta \cup N) = (B_\alpha \setminus B_\beta) \setminus N = \Phi, \text{ as } N = \bigcup(B_\alpha \setminus B_\beta).$$

Thus $B'_\alpha \subset B'_\beta$. By Lemma (2.6) there is a measurable function f such that $f \leq \gamma$ on B'_γ and $f \geq \gamma$ on $\tilde{B}'_\gamma$.

Let $\alpha \in D$, and choose a sequence $\langle \gamma_n \rangle$ from C with $\alpha < \gamma_n$ and $\alpha = \lim \gamma_n$. Then

$$B_\alpha \setminus B'_{\gamma_n} = B_\alpha \setminus (B_{\gamma_n} \cup N) \subset (B_\alpha \setminus B_{\gamma_n}).$$

Thus $P = \bigcup_n (B_\alpha \setminus B'_{\gamma_n})$ is a countable union of null sets and so a null set. Let $A = \bigcap B'_{\gamma_n}$. Then $f \leq \inf \gamma_n = \alpha$ on A, and $A \setminus B_\alpha \subset P$. Thus $f \leq \alpha$ almost everywhere on B_α. A similar argument shows that $f \geq \alpha$ almost everywhere on $\tilde{B}_\alpha$.

Let g be an extended real-valued function with $g \leq \gamma$ a.e. on B_γ and $g \geq \gamma$ on $\tilde{B}_\gamma$ for each $\gamma \in C$. Then $g \leq \gamma$ on B'_γ and $g \geq \gamma$ a.e. on $\tilde{B}'_\gamma$ except for x in a null set Q_γ. Thus $Q = \bigcup Q_\gamma$ is a null set and we must have $f = g$ on $X \setminus Q$.

3 Integration

Many definitions and proofs of Chapter 8 *Lebesgue Integral* depend on only those properties of Lebesgue measure which are also true for an arbitrary measure in an abstract measure space and carry over to this case.

(3.1) Definition (*Integral of a nonnegative simple function*) If E is a measurable set and φ a nonnegative simple function, we define

$$\int_E \varphi\, d\mu = \sum_{i=1}^{n} c_i \mu\big(E_i \cap E\big),$$

where

$$\varphi(x) = \sum_{i=1}^{n} c_i \chi_{E_i}(x).$$

It is easily seen that the value of this integral is independent of the representation of φ which we use. If a and b are positive numbers and φ and ψ nonnegative simple functions, then

$$\int a\varphi + b\psi = a\int \varphi + b\int \psi.$$

(3.2) Proposition Let f be a bounded measurable function which is identically zero outside a measurable set E of finite measure. Then

$$\inf\left\{\int_E \psi(x)dx : f \le \psi\right\} = \sup\left\{\int_E \varphi(x)dx : \varphi \le f\right\}$$

for all simple functions φ and ψ if and only if $f = g$ almost everywhere for some measurable function g.

Proof Proceed as in Proposition (8.14) of Chapter 8 *The Lebesgue Integral.*

(3.3) Definition (*Integral of a nonnegative simple function*) Let f be a nonnegative extended real-valued measurable function on the measure space $\big(X, \mathcal{B}, \mu\big)$. Then $\int f\, d\mu$ is the supremum of all integrals $\int \varphi\, d\mu$ as φ ranges over all simple functions with $0 \le \varphi \le f$.

i.e., $$\int f\, d\mu = \inf_{0 \le \varphi \le f} \int \varphi\, d\mu$$

- It follows immediately from this definition that $f \le g$ implies that $\int f \le \int g$ and $\int cf = c\int f$.

- We cannot immediately say that $\int f+g=\int f+\int g$, we need the Proposition (3.6).

(3.4) Theorem (Fatou's Lemma) If $\langle f_n\rangle$ is a sequence of nonnegative measurable functions that converge almost everywhere on a set E to a function f. Then

$$\int_E f \le \underline{\lim} \int_E f_n.$$

Proof

Without loss of generality, we may assume that $f_n(x)\to f(x)$ for each $x\in E$. From the definition of $\int f$ it suffices to show that, if φ is any nonnegative simple function with $\varphi\le f$, then $\int_E \varphi \le \underline{\lim} \int_E f_n$.

If $\int_E \varphi=\infty$, then there is a measurable set $A\subset E$ with $\mu A=\infty$ such that $\varphi>a>0$ on A. Set

$$A_n=\{x\in E:\, f_k(x)>a\ \forall k\ge n\}.$$

Then $\langle A_n\rangle$ is an increasing sequence of measurable sets whose union contains A, since $\varphi\le \lim f_n$. Thus $\mu A_n=\infty$. Since $\int_E f_n\ge a\mu A_n$, we have $\lim \int_E f_n=\infty=\int_E \varphi$.

If $\int_E \varphi<\infty$, then the set $A=\{x\in E:\varphi(x)>0\}$ is a measurable set of finite measure. Let M be the maximum of φ, let ε be a given positive number, and set

$$A_n=\{x\in E:\, f_k(x)>(1-\varepsilon)\varphi(x)\ \forall k\ge n\}.$$

Then $\langle A_n\rangle$ is an increasing sequence of sets whose union contains A, and so $\langle A\setminus A_n\rangle$ is a decreasing sequence of sets whose intersection is empty. By Proposition (1.7) $\lim \mu(A\setminus A_n)=0$, and so we can find an n such that $\mu(A\setminus A_k)<\varepsilon$ for all $k\ge n$. Thus for $k\ge n$

$$\begin{aligned}\int_E f_k &\ge \int_{A_k} f_k \ge (1-\varepsilon)\int_{A_k}\varphi\\ &\ge (1-\varepsilon)\int_E \varphi-\int_{A\setminus A_k}\varphi\\ &\ge \int_E \varphi-\varepsilon\left[\int_E \varphi+M\right].\end{aligned}$$

Hence

$$\underline{\lim}\int_E f_n \geq \int_E \varphi - \varepsilon\left[\int_E \varphi + M\right].$$

Since ε is arbitrary,

$$\underline{\lim}\int_E f_n \geq \int_E \varphi,$$

and this completes the proof.

(3.5) Monotone Convergence Theorem Let $\langle f_n \rangle$ be an increasing sequence of nonnegative measurable functions which converge almost everywhere to a function f and suppose that $f_n \leq f \ \forall n.$ Then

$$\int f = \lim \int f_n.$$

Proof

By Fatou's Lemma , we have

$$\int f \leq \underline{\lim}\int f_n. \tag{1}$$

But for each n we have $f_n \leq f$, and so $\int f_n \leq \int f$. But this implies

$$\overline{\lim}\int f_n \leq \int f \ . \tag{2}$$

Since $\underline{\lim}\int f_n \leq \overline{\lim}\int f_n$, (1) and (2) gives

$$\int f \leq \underline{\lim}\int f_n \leq \overline{\lim}\int f_n \leq \int f \ . \tag{3}$$

(3) tell us that $\underline{\lim}\int f_n = \overline{\lim}\int f_n = \int f$, and hence $\lim \int f_n$ exists and

$$\lim \int f_n = \int f.$$

(3.6) Proposition If f and g are nonnegative measurable functions and a and b nonnegative constants, then

$$\int af + bg = a\int f + b\int g.$$

We have

$$\int f \geq 0$$

with equality only if $f = 0$ a.e.

Proof

To prove the first statement, let $\langle \varphi_n \rangle$ and $\langle \psi_n \rangle$ be increasing sequences of simple functions which converge to f and g. Then $\langle a\varphi_n + b\psi_n \rangle$ is an increasing sequence of simple functions which converge to $af + bg$. By the Monotone Convergence Theorem, we have

$$\begin{aligned} \int af + bg &= \lim \int a\varphi_n + b\psi_n \\ &= \lim \left(a \int \varphi_n + b \int \psi_n \right) \\ &= a \int f + b \int g. \end{aligned}$$

Clearly $\int f \geq 0$. If $\int f = 0$, let $A_n = \left\{ x : f(x) \geq \frac{1}{n} \right\}$. Then we have $f \geq \frac{1}{n} \chi_{A_n}$, and so $\mu A_n = \int \chi_{A_n} = 0$. Since the set where $\int f > 0$ is the union of the sets A_n, it has measure zero.

(3.7) Corollary Let $\langle f_n \rangle$ be a sequence of nonnegative measurable functions. Then

$$\int \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int f_n.$$

(3.8) Definition A nonnegative function f is called **integrable** (over a measurable set E with respect to μ) if it is measurable and

$$\int f \, d\mu < \infty.$$

(3.9) Definition An arbitrary function f is said to be **integrable** if both f^+ and f^- are integrable. In this case we define

$$\int_E f = \int_E f^+ - \int_E f^-.$$

(3.10) Proposition If f and g are integrable functions and E is a measurable set, then

(i) $\int_E (c_1 f + c_2 g) = c_1 \int_E f + c_2 \int_E g.$

(ii) If $|h| \leq |f|$ and f is measurable then h is integrable.

(iii) If $f \geq g$ a.e., then $\int f \geq \int g$.

(3.11) Lebesgue Convergence Theorem Let g be integrable over E, and suppose that $\langle f_n \rangle$ is a sequence of measurable functions such that on E

$$|f_n(x)| \le g(x)$$

and such that almost everywhere on E

$$f_n(x) \to f(x).$$

Then

$$\int_E f = \lim \int_E f_n.$$

Proof

Apply Fatou's Lemma to sequences $g + f_n$ and $g - f_n$.

4 General Convergence Theorems

(4.1) Proposition Let $(X, \mathcal{B})$ be a measurable space, $\langle \mu_n \rangle$ a sequence of measures that converge setwise to a measure μ, and $\langle f_n \rangle$ a sequence of nonnegative measurable functions that converge pointwise to the function f. Then

$$\int f = \underline{\lim} \int f_n \, d\mu_n. \qquad \text{(UQ 2006)}$$

Proof

Setwise convergence of μ_n to μ implies that

$$\int \varphi \, d\mu = \lim \int \varphi \, d\mu_n.$$

for any simple function φ. From the definition of $\int f \, d\mu$ it suffices to prove that $\int \varphi \, d\mu \le \underline{\lim} \int f_n \, d\mu_n$ for any simple function $\varphi \le f$.

Suppose $\int \varphi \, d\mu < \infty$. Then φ vanishes outside a set E of finite measure. Let ε be a positive number, and set

$$E_n = \{x : f_k(x) \ge (1-\varepsilon)\varphi(x) \text{ for all } k \ge n\}.$$

Then $\langle E_n \rangle$ is an increasing sequence of sets whose union contains E, and so $\langle E \setminus E_n \rangle$ is a decreasing sequence of measurable sets whose intersection is empty. Thus by Proposition (1.7) there is an m such that $\mu(E \setminus E_m) < \varepsilon$. Since $\mu(E \setminus E_m) = \lim \mu_k(E \setminus E_m)$, we may

choose $n \geq m$ so that $\mu_k(E \setminus E_m) < \varepsilon$ for $k \geq n$. Since $E \setminus E_k \subset E \setminus E_m$, we have $\mu_k(E \setminus E_k) < \varepsilon$ for $k \geq n$. Thus

$$\int f_k d\mu_k \geq \int_{E_k} f_k d\mu_k \geq (1-\varepsilon) \int_{E_k} \varphi \, d\mu_k$$

$$\geq (1-\varepsilon) \int_E \varphi \, d\mu_k - \int_{E \setminus E_k} \varphi \, d\mu_k$$

$$\geq (1-\varepsilon) \int_E \varphi \, d\mu_k - M\varepsilon,$$

where M is the maximum of φ. Thus

$$\underline{\lim} \int f_k d\mu_k \geq \int_E \varphi \, d\mu - \varepsilon \left[M + \int \varphi \, d\mu \right].$$

Since ε was arbitrary,

$$\underline{\lim} \int f_k d\mu_k \geq \int_E \varphi \, d\mu .$$

i.e.,

$$\int_E \varphi \, d\mu \leq \underline{\lim} \int f_k d\mu_k .$$

The case when $\int_E \varphi \, d\mu = \infty$ is handled similarly.

(4.2) Proposition Let $(X, \mathcal{B})$ be a measurable space and $\langle \mu_n \rangle$ a sequence of measures on $\mathcal{B}$ that converge setwise to a measure μ. Let $\langle f_n \rangle$ and $\langle g_n \rangle$ be two sequences of measurable functions that converge pointwise to f and g. Suppose that $|f_n| \leq g_n$ and that

$$\lim \int g_n d\mu_n = \int g \, d\mu < \infty.$$

Then

$$\lim \int f_n d\mu_n = \int f \, d\mu.$$

Proof

Apply Proposition (4.1) to sequences $g_n + f_n$ and $g_n - f_n$.

5 Signed Measures

(5.1) Definition By a **signed measure** on the measurable space $(X, \mathcal{B})$ we mean an extended real-valued set function ν defined for the sets of $\mathcal{B}$ and satisfying the following conditions:

(i) v assumes at most one of the values $+\infty$, $-\infty$.

(ii) $v(\Phi) = 0$.

(iii) $v\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} vE_i$ for any sequence E_i of disjoint measurable sets, the equality taken to mean that the series on the right converges absolutely if $v\left(\bigcup_{i=1}^{\infty} E_i\right)$ is finite and that is properly diverges otherwise.

(5.2) Definition We say that a set A is a **positive set** with respect to a signed measure v if A is measurable and for every measurable subset E of A we have $vE \geq 0$.

- Every measurable subset of a positive set is again positive, and if we take the restriction of v to a positive set we obtain a measure.

(5.3) Definition We say that a set A is a **negative set** with respect to a signed measure v if A is measurable and for every measurable subset E of A we have $vE \leq 0$.

(5.4) Definition A set that is both positive and negative with respect to v is called a **null set.**

- A measurable set is a null set if and only if every measurable subset of it has v measure zero.

(5.5) Lemma Every measurable subset of a positive set is itself positive. The union of a countable collection of positive sets is positive. (UQ 2008)

Proof

The first statement is trivially true by the definition of a positive set. To prove the second statement, let A be the union of a sequence $\langle A_n \rangle$ of positive sets. If E is any measurable subset of A, set

$$E_n = E \cap A_n \cap \tilde{A}_{n-1} \cap \cdots \cap \tilde{A}_1.$$

Then E_n is a measurable subset of A_n and so $vE_n \geq 0$. Since the E_n are disjoint and $E = \bigcup E_n$, we have

$$vE = \sum_{n=1}^{\infty} vE_n \geq 0.$$

Thus A is a positive set.

$$E_n = E \cap A_n \cap \tilde{A}_{n-1} \cap \cdots \cap \tilde{A}_1.$$

(5.6) Lemma Let E be a measurable set such that $0 < vE < \infty$. Then there is a positive set A contained in E with $vA > 0$.

Proof

Either E itself is a positive set or it contains sets of negative measure. In the latter case let n_1 be the smallest positive integer such that there is a measurable set $E_1 \subset E$ with

$$vE_1 < -\frac{1}{n_1}.$$

Proceeding inductively, if $E \setminus \bigcup_{j=1}^{k-1} E_j$ is not already a positive set, let n_k be the smallest positive integer for which there is a measurable set E_k such that

$$E_k \subset E \setminus \left[\bigcup_{j=1}^{k-1} E_j \right]$$

and

$$vE_k < -\frac{1}{n_k}.$$

If we set

$$A = E \setminus \bigcup_{k=1}^{\infty} E_k,$$

then

$$E = A \cup \left[\bigcup_{k=1}^{\infty} E_k \right].$$

Since this is a disjoint union, we have

$$vE = vA + \sum_{k=1}^{\infty} vE_k$$

with the series on the right absolutely convergent, since vE is finite. Thus $\sum \frac{1}{n_k}$ converges, and we have $n_k \to \infty$. Since $vE_k \leq 0$ and $vE > 0$, we must have $vA > 0$.

To show that A is a positive set, let $\varepsilon > 0$ be given. Since $n_k \to \infty$, we may choose k so large that $(n_k - 1)^{-1} < \varepsilon$. Since

$$A \subset E \setminus \cup \left[\bigcup_{j=1}^{\infty} E_j \right],$$

A can contain no measurable sets with measure less than $-(n_k - 1)^{-1}$, which is greater than $-\varepsilon$. Thus A contains no measurable sets of measure less than $-\varepsilon$. Since ε is an arbitrary positive number, it follows that A can contain no sets of negative measure and so must be a positive set.

(5.7) Proposition (Han Decomposition Theorem) Let ν be a signed measure on the measurable space $(X, \mathcal{B})$. Then there is a positive set A and a negative set B such that $X = A \cup B$ and $A \cap B = \Phi$.

Proof

Without loss of generality we may assume that $+\infty$ is the infinite value omitted by ν. Let λ be the supremum of νA over all sets A that are positive with respect to ν. Since the empty set is positive, $\lambda \geq 0$. Let $\langle A_i \rangle$ be a sequence of positive sets such that

$$\lambda = \lim_{i \to \infty} \nu A_i,$$

and set

$$A = \bigcup_{i=1}^{\infty} A_i.$$

By Lemma (5.5) the set A is itself a positive set, and so $\lambda \geq \nu A$. But $A \setminus A_i \subset A$ and so $\nu(A \setminus A_i) \geq 0$. Thus

$$\nu A = \nu A_i + \nu(A \setminus A_i) \geq \nu A_i.$$

Hence $\nu A \geq \lambda$, and so $\nu A = \lambda$, and $\lambda < \infty$.

Let $B = \tilde{A}$, the complement of A, and suppose that E is a positive subset of B. Then E and A are disjoint and $E \cup A$ is a positive set. Hence

$$\lambda \geq \nu(E \cup A) = \nu E + \nu A = \nu E + \lambda,$$

whence $\nu E = 0$, since $0 \leq \lambda < \infty$. Thus B contains no positive subsets of positive measure and hence no susbsets of positive measure by Lemma (5.6). Consequently, B is a negative set.

(5.8) Definition A decomposition of X into two disjoint sets A and B such that A is positive for ν and B negative is called a **Hahn decomposition** for ν. Proposition (5.7) states the existence of a Hahn decomposition for each signed measure. Hahn decomposition is not unique.

(5.9) Definition Two measures v_1 and v_2 on $(X, \mathcal{B})$ are said to be mutually singular (in symbols $v_1 \perp v_2$) if there are disjoint measurable sets A and B with $X = A \cup B$ such that $v_1(A) = v_2(B) = 0.$

(5.10) Proposition Let v be a signed measure on the measurable space $(X, \mathcal{B})$. Then there are two mutually singular measures v^+ and v^- on $(X, \mathcal{B})$ such that $v = v^+ - v^-$. Moreover, there is only one such pair of mutually singular measures.

Proof

By Proposition (5.7), the signed measure v has a Hahn decomposition, say $\{A, B\}$ with $X = A \cup B$ and $A \cap B = \Phi.$

We define two measures v^+ and v^- on $(X, \mathcal{B})$ with $v = v^+ - v^-$ by setting

$$v^+(E) = v(E \cap A)$$

and

$$v^-(E) = -v(E \cap B).$$

The measures v^+ and v^- on $(X, \mathcal{B})$ are mutually singular because $v^+(A) = v(A \cap A) = v(\Phi) = 0$ and $v^-(A) = -v(A \cap A) = -v(\Phi) = 0$ on the measurable sets A and B that are disjoint and with $X = A \cup B$.

The uniqueness part is left to the exercise.

(5.11) Definition The decomposition of v given by the Proposition (5.10) is called the **Jordan decomposition** of v. The measures v^+ and v^- are called the positive and negative parts (or variations) of v. Since v assumes at most one of the values $+\infty$ and $-\infty$, either v^+ and v^- must be finite. If they are both finite, we call v a **finite signed measure.**

The measure $|v|$ defined by

$$|v|(E) = v^+E + v^-E$$

is called the **absolute value** or **total variation** of v. A set E is *positive* for v if $v^-E = 0$. It is a *null set* if $|v|(E) = 0.$

6 The Radon-Nikodym Theorem

(6.1) Definition Let $(X, \mathcal{B})$ be a fixed measurable space. If μ and ν are two measures defined on $(X, \mathcal{B})$, we say that μ and ν are **mutually singular** (and write $\mu \perp \nu$) if there are disjoint sets A and B in $\mathcal{B}$ such that $X = A \cup B$ and $\nu A = \mu B = 0$.

Despite the fact that the notion of singularity is symmetric in ν and μ, we some times say that ν is singularity with respect to μ.

(6.2) Definition A measure ν is said to be **absolutely continuous** with respect to the measure μ if $\nu A = 0$ for each set A for which $\mu A = 0$. We use the symbolism $\nu \ll \mu$ for ν absolutely continuous with respect to μ.

- In the case of signed measures μ and ν, we say $\nu \ll \mu$ if $|\nu| \ll |\mu|$ and $\nu \perp \mu$ if $|\nu| \perp |\mu|$.
- Whenever we are dealing with more than one measure, we must specify almost everywhere with respect to the given measure, say μ and it is abbreviated as a.e.$[\mu]$.
- If $\nu \ll \mu$ and a property holds a.e. $[\mu]$, then it holds for a.e. $[\nu]$.

Let μ be a measure and f a nonnegative measurable function on X. For E in $\mathcal{B}$, set

$$\nu E = \int_E f \, d\mu.$$

Then ν is a **set function** defined on $\mathcal{B}$.

- It follows from Corollary (3.7) that ν is countably additive and hence a *measure.*
- The measure ν will be finite if and only if f is integrable.
- Since the integral over a set of μ-measure zero is zero, we have ν absolutely continuous with respect to μ.

(6.3) Radon-Nidkodym Theorem Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space, and let ν be a measure defined on $\mathcal{B}$ which is absolutely continuous with respect to μ. Then there is a nonnegative measurable function f such that for each set E in $\mathcal{B}$ we have

$$\nu E = \int_E f \, d\mu.$$

The function f is unique in the sense that if g is any measurable function with this property then $g = f$ a.e. $[\mu]$.

Proof

The extension from the finite to the σ-finite case is not difficult and is left to the student. Thus we shall assume that μ is finite. Then $\nu - \alpha\mu$ is a signed measure for each rational number α. Let (A_α, B_α) be a Hahn decomposition for $\nu - \alpha\mu$, and take $A_0 = X$, $B_0 = \Phi$.

Now $B_\alpha \setminus B_\beta = B_\alpha \cap A_\beta$. Thus $(\nu - \alpha\mu)(B_\alpha \setminus B_\beta) \le 0$, and hence $(\nu - \beta\mu)(B_\alpha \setminus B_\beta) \ge 0$. If $\beta > \alpha$, these imply $\mu(B_\alpha \setminus B_\beta) = 0$, and so by Proposition (2.7) there is a measurable function f such that for each rational α we have $f \ge \alpha$ a.e. on A_α and $f \le \alpha$ a.e. on B_α. Since $B_0 = \Phi$, we may take f to be nonnegative.

Let E be an arbitrary set in $\mathcal{B}$, and set

$$E_k = E \cap \left(B_{\frac{k+1}{N}} \setminus B_{\frac{k}{N}}\right), \qquad E_\infty = E \setminus \bigcup B_{\frac{k}{N}}.$$

Then $E = E_\infty \cup \bigcup_{k=0}^{\infty} E_k$, and this union is disjoint modulo null sets. Thus

$$\nu E = \nu E_\infty + \sum_{k=0}^{\infty} \nu E_k.$$

Since $E_k \subset B_{\frac{k+1}{N}} \cap A_{\frac{k}{N}}$, we have $\frac{k}{N} \le f \le \frac{k+1}{N}$ on E_k, and so

$$\tfrac{k}{N}\mu E_k \le \int_{E_k} f \, d\mu \le \tfrac{k+1}{N}\mu E_k. \tag{1}$$

Since $\tfrac{k}{N}\mu E_k \le \nu E_k \le \tfrac{k+1}{N}\mu E_k$,

we have

$$\nu E_k - \frac{1}{N}\mu E_k \le \tfrac{k+1}{N}\mu E_k - \frac{1}{N}\mu E_k = \tfrac{k}{N}\mu E_k \le \int_{E_k} f \, d\mu \le \tfrac{k}{N}\mu E_k + \tfrac{1}{N}\mu E_k \le \nu E_k + \tfrac{1}{N}\mu E_k$$

i.e., $$\nu E_k - \frac{1}{N}\mu E_k \le \int_{E_k} f \, d\mu \le \nu E_k + \frac{1}{N}\mu E_k.$$

On E_∞ we have $f = \infty$ a.e. If $\mu E_\infty > 0$, we must have $\mu E_\infty = \infty$, since $(\nu - \alpha\mu)E_\infty$ is positive for each α. If $\mu E_\infty = 0$, we have $\nu E_\infty = 0$, since $\nu \ll \mu$. In either case

$$\nu E_\infty = \int_{E_\infty} f \, d\mu.$$

Adding this equality and our previous inequalities gives

$$\nu E - \frac{1}{N}\mu E \le \int_E f \, d\mu \le \nu E + \frac{1}{N}\mu E.$$

Since μE is finite and N arbitrary, we must have

$$\nu E = \int_E f \, d\mu.$$

(6.4) Definition The function f given by Radon-Nidkodym Theorem is called **Radon-Nidkodym derivative of** ν with respect to μ. It is denoted by $\left[\frac{d\nu}{d\mu}\right]$.

(6.5) Proposition (Lebesgue Decomposition) Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space [Ref. Definition (1.11)] and ν a σ-finite measure defined on $\mathcal{B}$. Then we can find a measure ν_0, singular with respect to μ, a measure ν_1, absolutely continuous with respect to μ, such that $\nu = \nu_0 + \nu_1$. The measures ν_0 and ν_1 are unique.

Proof

We prove the existence of ν_0 and ν_1, leaving the uniqueness part to the exercise.

Since μ and ν are σ-finite measures, so is the measure $\lambda = \mu + \nu$. Since both μ and ν are absolutely continuous with respect to λ, the Radon-Nidkodym Theorem asserts the existence of nonnegative masurable functions f and g such that for each $E \in \mathcal{B}$

$$\mu E = \int_E f \, d\lambda, \qquad \nu E = \int_E g \, d\lambda.$$

Let $A = \{x : f(x) > 0\}$ and $B = \{x : f(x) = 0\}$. Then X is the disjoint union of A and B, $\mu B = 0$. If we define ν_0 by

$$\nu_0 E = \nu(E \cap B),$$

we have $\nu_0(A) = 0$ and so $\nu_0 \perp \mu$. Let

$$\nu_1 E = \nu(E \cap A) = \int_{E \cap A} g \, d\lambda.$$

Then $\nu = \nu_0 + \nu_1$, and we have only to show that $\nu_1 \ll \mu$. Let E be a set of μ measure zero. Then

$$0 = \mu E = \int_E f \, d\lambda,$$

and $f = 0$ a.e. $[\lambda]$ on E. Since $f > 0$ on $A \cap E$, we must have $\lambda(A \cap E) = 0$. Hence $\nu(A \cap E) = 0$, and so $\nu_1(E) = \nu(A \cap E) = 0$.

M.Sc. (PREVIOUS) DEGREE EXAMINATION, MAY 2008

Mathematics

Paper III – REAL ANALYSIS

(2000 Admission Onwards)

(Regular – For SDE candidates)

Time: Three Hours Maximum: 120 Marks

Answer **all** *questions from Part A.*

Each question carries 4 *marks.*

Answer any **four** *questions from Part B without omitting any Unit.*

Each question carries 24 marks.

Part A

1. (*i*) Describe the least upper bound property of ordered sets. Verify whether the ordered field $\square$ of rationals have this property.

   (*ii*) Let Y be the unit circle and let $f:[0, 2\pi) \to Y$ be defined by $f(t) = (\cos t, \sin t)$. Verify that f is one-to-one and onto.

   (*iii*) Let f be defined on $[a, b]$ with $m \le f(x) \le M$ for all x. With the usual notations prove that $m(b-a) \le L(P, f) \le U(P, f) \le M(b-a)$,

   (*iv*) Prove that Lebesgue measure of a singleton set is zero.

   (*v*) If $a_1\chi_{E_1} + a_2\chi_{E_2} = b_1\chi_{F_1} + b_2\chi_{F_2}$ then is it necessary that $a_1 = b_1$ and $a_2 = b_2$? Justify you answer.

   (*vi*) Give an example of a measure μ which is different from the Lebesgue measure m and satisfying $\mu << m$.

($6 \times 4 = 24$ marks)

Part B

UNIT I

2. (*a*) Define neighborhood of a point in a metric space. Show that every neighborhood is an open set.

   (*b*) Show that if p is a limit point of a set E then every neighborhood of p contains infinitely many points of E.

3. (*a*) Let X, Y, Z be metric spaces, $f : X \to Y$ and $g : Y \to Z$. Show that if f is continuous at p and g is continuous at $f(p)$ then $h(x) = f(g(x))$ is continuous at p.

   (*b*) Show that $f : X \to Y$ is continuous at every point at x if and only if $f^{-1}(V)$ is open in X whenever V is open in Y.

4. (*a*) Let f be defined on $[a, b]$ and suppose that f has a local maximum at $x \in (a, b)$. Show that if $f'(x)$ exists then $f'(x) = 0$.

   (*b*) Let f be continuous on $[a, b]$ and differentiable on (a, b). Show that there exists $x \in (a, b)$ such that $f(b) - f(a) = (b - a) f'(x)$.

UNIT II

5. (*a*) Let α be increasing on $[a, b]$. Show that $f \in \Re(\alpha)$ on $[a, b]$ if and only if for every $\varepsilon > 0$ there exists a partition P such that $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$.

   (*b*) Show that if f is continuous on $[a, b]$ and α is increasing on $[a, b]$ then $f \in \Re(\alpha)$ on $[a, b]$.

6. (*a*) Define equicontinuous family of functions.

   (*b*) Let K be a compact metric space and $f_n \in \wp(K)$ for $n = 1, 2, 3, \ldots$ show that:

   (*i*) if (f_n) is uniformly convergent in K then (f_n) is equicontinuous.

   (*ii*) if (f_n) is equicontinous and point wise bounded then (f_n) contains a uniformly convergent subsequence.

7. (*a*) Let $\langle A_n \rangle$ be a sequence of sets of real numbers and m^* be the Lebesgue outer measure. Show that $m^*\left(\bigcup A_n\right) \leq \sum m^* A_n$.

   (*b*) Prove that if A is countable, then $m^* A = 0$.

UNIT III

8. (*a*) State and prove Fatou's Lemma.

   (*b*) State and prove Lebesgue convergence theorem.

9. Let f be increasing real valued function on $[a, b]$. Show that

   (a) f is differentiable a.e. on $[a, b]$.

   (b) the derivative f' is measurable.

10. (a) Define positive set with respect to a signed measure ν. Show that if E is measurable and $0 < \nu E < \infty$, then E contains a positive set A with $\nu E > 0$.

    (b) Prove that the union of a countable collection of positive sets is positive.

($4 \times 24 = 96$ marks)

M.Sc. (PREVIOUS) DEGREE EXAMINATION, MARCH/APRIL 2006

Mathematics

Paper III – REAL ANALYSIS

(2000 Admission Onwards)

(Regular – For SDE candidates)

Time: Three Hours Maximum: 120 Marks

Answer **all** *questions from Part A.*

Each question carries 4 *marks.*

Answer any **four** *questions from Part B without omitting any Unit.*

Each question carries 24 marks.

Part A

1. (i) Prove that there is no rational number whose square is 12.

   (ii) Assume that f is a continuous real function defined in (a, b) such that $f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2}$ for all $x, y \in (a, b)$. Prove that f is convex.

   (iii) Evaluate $\int_0^5 x\, d\left([x]-x\right)$ where $[x]$ denotes the largest integer not greater than x.

   (iv) Show that canotor ternary set has measure zero.

(*v*) Let $\varphi = \sum_{i=1}^{n} a_i \chi_{E_i}$, with $E_i \cap E_j = \Phi$ for $i \neq j$. Suppose each set E_i is a measurable set of finite measure. Show that $\int \varphi = \sum_{i=1}^{n} a_i m E_i$.

(*vi*) Prove that $f(x) = \begin{cases} 2, & x \leq 0 \\ x^2, & x > 0 \end{cases}$ is measurable.

(6 × 4 = 24 marks)

Part B

UNIT I

2. (*a*) For every real $x > 0$ and every integer $n > 0$ prove that there is one and only one real y such that $y^n = x$. (8 marks)

(*b*) If $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are complex numbers, then show that

$$\left| \sum_{j=1}^{n} a_j \overline{b_j} \right|^2 \leq \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2.$$ (8 marks)

(*c*) Suppose $\mathbf{a} \in \mathbb{R}^k$, $\mathbf{b} \in \mathbb{R}^k$. Find $\mathbf{c} \in \mathbb{R}^k$ and $r > 0$ such that $|\mathbf{x} - \mathbf{a}| = 2|\mathbf{x} - \mathbf{b}|$ if and only if $|\mathbf{x} - \mathbf{c}| = r$. (8 marks)

3. (*a*) Let f be a continuous mapping of a compact metric space *X* into a metric space *Y*. Show that f is uniformly continuous on *X*. (8 marks)

(*b*) Let f be monotonically increasing on (a,b). Show that $f(x+)$ and $f(x-)$ exist at every point $x \in (a,b)$. Further show that

$$\sup_{a<t<x} f(t) = f(x-) \leq f(x) \leq f(x+) = \inf_{a<t<x} f(t)$$

and if $a < x < y < b$, then $f(x+) \leq f(y-)$. (8 + 4 =12 marks)

(*c*) Let f be monotonic on (a,b). Then the set of points of (a,b) at which f is discontinuous is at most countable. (4 marks)

4. (*a*) State and prove Taylor's theorem in $\mathbb{R}^1$. (8 marks)

(*b*) Suppose $\mathbf{f}$ is a continuous mapping of $[a, b]$ into $\mathbb{R}^k$ and $\mathbf{f}$ is differentiable in (a,b). Then there exists $x \in (a, b)$ such that $|\mathbf{f}(b) - \mathbf{f}(a)| \leq (b-a)|\mathbf{f}'(x)|$.

(8 marks)

(*c*) Suppose f is defined in a neighborhood of x, and suppose $f''(x)$ exists. Show that $\lim_{h\to 0}\frac{f(x+h)+f(x-h)-2f(x)}{h^2}=f''(x)$. Show by an example that the limit may exist even if $f''(x)$ does not. (8 marks)

UNIT II

5. (*a*) Show that $f\in\Re(\alpha)$ on $[a,b]$ if and only if for every $\varepsilon>0$ there exists a partition P such that $U(P,f,\alpha)-L(P,f,\alpha)<\varepsilon$. (12 marks)

(*b*) Assume α increases monotonically and $\alpha'\in\Re$. Let f be a bounded real function on $[a,b]$. Then show that $f\in\Re(\alpha)$ if and only if $f\alpha'\in\Re$. In that case

$$\int_a^b f\,d\alpha=\int_a^b f(x)\alpha'(x)dx.$$ (12 marks)

6. (*a*)]Suppose $f_n\to f$ uniformly on a set E in a metric space. Let x be a limit point of E, and suppose that $\lim_{t\to x} f_n(t)=A_n\ (n=1,2,3,\ldots)$. Then show that $\{A_n\}$ converges, and $\lim_{t\to x} f(t)=\lim_{n\to\infty} A_n$. (12 marks)

(*b*) Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a, b]$ and such that $\{f_n(x_0)\}$ converges from some point $x_0\in[a, b]$. If $\{f_n'\}$ converges uniformly on $[a, b]$, then $\{f_n\}$ converges uniformly on $[a, b]$, to a function f, and $f'(x)=\lim_{n\to\infty} f_n'(x)\ (a\le x\le b)$. (12 marks)

7. (*a*) If K is compact, if $f_n\in\wp(K)$ for $n=1,2,3,\ldots$, and if $\{f_n\}$ is pointwise bounded and equicontinuous on K, then show that

(i) $\{f_n\}$ is uniformly bounded on K.

(ii) $\{f_n\}$ contains a uniformly convergent subsequence. (12 marks)

(*b*) Prove that every Borel set is measurable. (6 marks)

(*c*) Construct a non-measurable set. (6 marks)

UNIT III

8. (*a*) State and prove Lebesgue (dominated) convergence theorem. Is the condition "dominated" necessary. Justify your answer. (12 marks)

(*b*) Let $\{f_n\}$ be a sequence of measurable functions that converges in measure to f. Show that there is a subsequence $\{f_{n_k}\}$ that converges to f almost everywhere.

(12 marks)

9. (*a*) Let f be an increasing real-valued function on the interval $[a,b]$.Prove that f is differentiable almost everywhere and the derivative f' is measurable, and $\int_a^b f'(x)dx \le f(b) - f(a)$. (12 marks)

(*b*) If f is an integrable function on $[a, b]$, show that F defined on $[a, b]$ by $F(x) = \int_a^x f(t)\,dt$ is a continuous function of bounded variation on $[a, b]$.

(12 marks)

10. (*a*) If f is bounded and measurable on $[a, b]$ and $F(x) = \int_a^x f(t)\,dt + F(a)$, then show that $F'(x) = f(x)$ for almost all x in $[a, b]$. (12 marks)

(*b*) Let $(X, \mathcal{B})$ be a measurable space, $\langle \mu_n \rangle$ a sequence of measures that converge setwise to a measure μ, and $\langle f_n \rangle$ a sequence of nonnegative measurable functions that converge pointwise to the function f. Prove that $\int f = \underline{\lim} \int f_n \, d\mu_n$.

(12 marks)